Solutionnaire de l'examen 2

1. On a $^t I(\varphi_0)(\sum a_n x^n) = \varphi_0 \circ I(\sum a_n x^n)$

$$= \varphi_0(\sum a_n x^{n+1}) = 0.$$

et si $k \geq 1$

$$^t I(\varphi_k)(\sum a_n x^n) = \varphi_k \circ I(\sum a_n x^n)$$

$$= \varphi_k(\sum a_n x^{n+1}) = a_{k-1} = \varphi_k(\sum a_n x^n)$$

car le coefficient de x^k dans $\sum a_n x^{n+1}$ est a_{k-1}.

2. On a

$$v \in \mathrm{Ker}(f) \iff v \in \mathrm{Ker}(f)^{\circ\circ}$$

$$\iff v \in \mathrm{Im}(^t f)^\circ$$

$$\iff \forall \psi \in \mathrm{Im}(^t f),\ \psi(v) = 0$$

$$\iff \forall \varphi \in W^*,\ ^t f(\varphi)(v) = 0$$

$$\iff \forall \varphi \in W^*,\ v \in \mathrm{Ker}(^t f(\varphi))$$

3. On a $\Theta(e_j^* \otimes e_i)(e_j) = e_j^*(e_j)\, e_i = e_i$

et $\Theta(e_j^* \otimes e_i)(e_{j'}) = e_j^*(e_{j'})\, e_i = 0$ si $j' \neq j$.

Donc la matrice de $\Theta(e_j^* \otimes e_i)$ est la matrice élémentaire E_{ij} $\Rightarrow$ $\mathrm{Tr}(\Theta(e_j^* \otimes e_i))$

$$= \mathrm{Tr}(E_{ij}) = \delta_{ij}.$$

On $\varphi \otimes v = \sum_{i,j} \beta_j \alpha_i\, e_j^* \otimes e_i$

$$\Rightarrow \Theta(\varphi \otimes v) = \sum_{i,j} \beta_j \alpha_i\, \Theta(e_j^* \otimes e_i)$$

$$\Rightarrow \mathrm{Tr}(\Theta(\varphi \otimes v)) = \sum_{i,j} \beta_j \alpha_i\, \mathrm{Tr}(\Theta(e_j^* \otimes e_i))$$

$$= \sum_{i,j} \beta_j \alpha_i \delta_{ij} = \sum_i \beta_i \alpha_i.$$

On a aussi

$$\varphi(v) = \Big(\sum_j \beta_j e_j^*\Big)\Big(\sum_i \alpha_i e_i\Big)$$

$$= \sum_{i,j} \beta_j \alpha_i e_j^*(e_i) = \sum_{i,j} \beta_j \alpha_i \delta_{ij}$$

$$= \sum_i \alpha_{ii} = \mathrm{Tr}\,(\Theta(\varphi \otimes v)) .$$

4. Par récurrence sur k :

$k = 0$ rien à démontrer

$k \geq 1$ On suppose que

$$x \wedge y_1 \wedge \cdots \wedge y_k = (-1)^k y_1 \wedge \cdots \wedge y_k \wedge x$$

Alors $x \wedge y_1 \wedge \cdots \wedge y_{k+1} = x \wedge y_1 \wedge \cdots \wedge y_k \wedge y_{k+1}$

$$= ((-1)^k y_1 \wedge \cdots \wedge y_k \wedge x) \wedge y_{k+1}$$

$$= (-1)^k y_1 \wedge \cdots \wedge y_k \wedge x \wedge y_{k+1}$$

$$= (-1)^k (- y_1 \wedge \cdots \wedge y_k \wedge y_{k+1} \wedge x)$$

$$= (-1)^{k+1} y_1 \wedge \cdots \wedge y_{k+1} \wedge x .$$

<u>Bonus</u> : $\Theta(\varphi \otimes v) \circ \Theta(\psi \otimes u)(w)$

$$= \Theta(\varphi \otimes v)(\Theta(\psi \otimes u)(w))$$

$$= \Theta(\varphi \otimes v)(\psi(w) u)$$

$$= \psi(w)\, \Theta(\varphi \otimes v)(u) = \psi(w)\, \varphi(u)\, v$$

$$= \varphi(u)\, \Theta(\psi \otimes v)(w)$$

Donc $\Theta(\varphi \otimes v) \circ \Theta(\psi \otimes u) = \varphi(u)\, \Theta(\psi \otimes v)$