# Higher Lie idempotents

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## 1 Introduction

Let $T(X)$ be the tensor bialgebra over an alphabet $X$. It is a graded connected cocommutative bialgebra, canonically isomorphic to the envelopping bialgebra of the free Lie algebra over $X$, $Lie(X)$. The subalgebra of its convolution algebra generated by the projections arising from the graduation is also an algebra for the composition of morphisms and is anti-isomorphic as such with the direct sum of the Solomon's symmetric groups descent algebras [R2]. For example, the canonical projections arising from the isomorphism between $T(X)$ and the envelopping bialgebra of $Lie(X)$ belong to this convolution algebra and may be identified with certain idempotents of the symmetric groups algebras (see [GR] [R2]). More generally, over a field of caracteristic zero, any connected graded cocommutative bialgebra $A$ is canonically isomorphic to the envelopping bialgebra of the Lie algebra of its primitive elements (Cartier-Milnor-Moore theorem, see [MM]). As for the tensor algebra, this isomorphism has a combinatorial description and the associated canonical projections lie in a suitable descent algebra [P1]. Moreover these results yield an effective computation of the inverse map to the canonical map from the envelopping bialgebra of the Lie algebra of primitive elements to $A$ and, as a corollary, a new combinatorial proof of the Cartier-Milnor-Moore theorem [P1]. All these results dualize automatically into properties of the shuffle (or cotensor) algebra over an alphabet and on connected commutative bialgebras.

The Cartier-Milnor-Moore theorem has a weak form, known as the Leray theorem (see [MM] Th.7.5). In the setting of connected graded commutative bialgebras, it asserts that any section of the projection from the augmentation ideal $I$ of such a bialgebra $A$ to the vector space of indecomposables $I/I^2$ induces an isomorphism between the free commutative algebra over $I/I^2$ and $A$. The dual statement states that any retract of the inclusion of the vector space of primitive elements $Prim(A)$ into the augmentation ideal of a connected cocommutative graded bialgebra $A$ induces an isomorphism between the cofree cocommutative coalgebra over $Prim(A)$ and $A$. When $A$ is the tensor algebra, such a retract is called a *Lie idempotent* (at least if it can be computed in the group algebras of the symmetric groups, naturally acting on the tensor algebra [R2]).

These idempotents have a long history. We are interested in this paper in their combinatorics in relation to the structure of bialgebras. Given a familly of Lie idempotents, using their convolution products, we construct and study families of idempotents whose properties yield an effective proof of the dual Leray theorem. In particular, we define three famillies of idempotents which all generalize those of [GR] and reduce to them when one chooses as Lie idempotent the canonical one in the tensor bialgebra $T(X)$. The second one is the most interesting from our point of view, since it describes the combinatorics underlying the dual Leray theorem. The last one generalizes a construction by Krob, Leclerc and Thibon [KLT]; our approach to these idempotents is different from theirs and emphasizes on the combinatorics of the tensor bialgebras.

## 2 Lie idempotents and their convolution algebras

We are working in all that follows with connected graded cocommutative bialgebras (our statements extend by duality easily to statements on connected graded commutative bialgebras).

Let $F$ be a field of characteristic 0 and $A = \bigoplus_{n=0}^{\infty} A_n$ a connected graded cocommutative bialgebra over $F$. The product of $A$ (resp. the coproduct, the unit, the counit) is written $\Pi$ (resp. $\Delta$, $\eta$, $\epsilon$). Let $\mathcal{L}(A)$ be the set of (degree 0) linear *graded* endomorphisms of $A$. The convolution product $f * g$ of two elements $f, g$ of $\mathcal{L}(A)$ is defined by:

$$f * g := \Pi \circ (f \otimes g) \circ \Delta.$$

For this product, $\mathcal{L}(A)$ is an associative algebra with unit $\eta \circ \epsilon$, called the *convolution algebra of* $A$ (see e.g. [B], [K] or [P1] for more details).

A *composition* $\lambda$ of $n \geq 0$ of *length* $l(\lambda) = k$ is a $k$-tuple $\lambda = (\lambda_1, ..., \lambda_k)$ of strictly positive integers whose *weight* $|\lambda| := \lambda_1 + ... + \lambda_k$ is equal to $n$. A *partition* is a weakly increasing composition. To each composition $\lambda$ is associated a unique partition $p(\lambda)$ by reodering the sequence $\lambda$ into increasing order.

**Definition 2.1** *For a given connected graded cocommutative bialgebra $A$, we call Lie idempotent any element of $\mathcal{L}(A)$ which is idempotent for the composition of endomorphisms and whose image is $Prim(A)$ .*

In other words, a Lie idempotent $\iota$ is a left inverse of the canonical injection

$$j : \ Prim(A) \hookrightarrow A.$$

When $A$ is the tensor algebra, this definition extends the classical definition of Lie idempotents [R2] .

Let $p_n \in \mathcal{L}(A)$ be the canonical projection on $A_n$, the degree $n$ component of $A$. The subalgebra $\mathcal{D}(A)$ of the convolution algebra of $A$ generated by the familly $(p_n)_{n\in\mathbf{N}}$ is, by definition, the *descent algebra* of $A$. If $\iota$ is a Lie idempotent, we set:

$$\iota_n := p_n \circ \iota = \iota \circ p_n = p_n \circ \iota \circ p_n$$

for $n \geq 0$. Note that these equalities hold because $\iota$ is homogeneous and preserves the grading. Note also that $\iota_0 = 0$, since $A_0 = F$.

**Definition 2.2** *If $\iota$ is a Lie idempotent for $A$, the $\iota$-descent algebra $\mathcal{D}_\iota$ of $A$ is the subalgebra of the convolution algebra of $A$ generated by the familly $(\iota_n)_{n\in\mathbf{N}}$.*

For any composition $\lambda$ of length $k$, let us set:

$$\iota_\lambda := \iota_{\lambda_1} * ... * \iota_{\lambda_k}.$$

The convolution product is homogeneous; hence $\iota_\lambda = p_{|\lambda|} \circ \iota_\lambda = \iota_\lambda \circ p_{|\lambda|}$. Thus the $\iota$-descent algebra decomposes as a direct sum

$$\mathcal{D}_\iota = \bigoplus_{n=0}^{\infty} \mathcal{D}_{\iota n}.$$

Indeed, any element $f$ of $\mathcal{D}_\iota$ is the sum of its restrictions $f_n := p_n \circ f \circ p_n$ to the components $A_n$ of $A$. Observe that, by the very definition of $\mathcal{D}_\iota$, the familly $\{\iota_\lambda\}_{|\lambda|=n}$ spans linearly $\mathcal{D}_{\iota n}$.

**Example** Let $e$ be the *canonical projection* on $Prim(A)$ (by definition, $e$ is the logarithm of the identity in the convolution algebra of $A$ [R2] [GR] [P1]). Then, $\mathcal{D}_e$ identifies with the descent algebra of $A$, $\mathcal{D}(A)$. When $A = T(X)$, the tensor algebra over an infinite alphabet $X$ (equivalently, the algebra of noncommutative polynomials on $X$), then the $e_\lambda$'s are linearly independent and form a basis of $\mathcal{D}(A)$[R2].

# 3 Higher idempotents of the first kind

Let us begin with some observations. First of all, $\iota_n \circ \iota_m = \delta_{mn}\iota_n$, by the very definition of $\iota$. Moreover, since $\iota$ maps $A$ onto its primitive part, we have for any $x \in A$:

$$\Delta \circ \iota_n(x) = \iota_n(x) \otimes 1 + 1 \otimes \iota_n(x).$$

Observe also that we have for all $\lambda, \mu$ such that $|\lambda| \neq |\mu|$ : $\iota_\lambda \circ \iota_\mu = 0$.

**Lemma 3.1** *Let $\mu$, $\lambda$ be two compositions of the same weight and the same length $k$.*

*(i) If $p(\lambda) \neq p(\mu)$, then $\iota_\mu \circ \iota_\lambda = 0$.*

*(ii) If $p(\lambda) = p(\mu)$, then $\iota_\mu \circ \iota_\lambda = N\iota_\mu$, where $N$ is the number of permutations of $\{1, ..., k\}$ which act trivially on the sequence $p(\mu) = p(\lambda)$.*

**Proof**

Let $\Delta_k$ (resp. $\Pi_k$) be the iterated coproduct (resp. product) map from $A$ to $A^{\otimes k}$ (resp $A^{\otimes k}$ to $A$). More generally, for any $l \geq 2$ and $k \geq 3$, let $\Delta_{l2}$ be the coproduct of the coalgebra $A^{\otimes l}$ and $\Delta_{lk}$ the associated iterated coproduct from $A^{\otimes l}$ to $(A^{\otimes l})^{\otimes k}$ (so that $\Delta_k = \Delta_{1k}$). All we need is the definition of $\Delta_{kk}$, which is defined by $\Delta_{kk} = \phi \circ \Delta_k^{\otimes k}$, where $\phi$ is defined, using tensors indexed by $k$ by $k$ matrices, by

$$\phi(x_{11} \otimes x_{12} \otimes ... \otimes x_{1k} \otimes x_{21} \otimes x_{22} \otimes ... \otimes x_{2k} \otimes ... \otimes x_{kk})$$

$$= x_{11} \otimes x_{21} \otimes ... \otimes x_{k1} \otimes x_{12} \otimes x_{22} \otimes ... \otimes x_{k2} \otimes ... \otimes x_{kk}.$$

Now, by the very definition of $\iota_\mu$ and $\iota_\lambda$:

$$\iota_\mu \circ \iota_\lambda = \Pi_k \circ (\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \Delta_k \circ \Pi_k \circ (\iota_{\lambda_1} \otimes ... \otimes \iota_{\lambda_k}) \circ \Delta_k.$$

It follows from the general properties of Hopf algebras that $\Delta_k \circ \Pi_k = [\Pi_k]^{\otimes k} \circ \Delta_{kk}$ (see e.g. [P2]), so that:

$$\iota_\mu \circ \iota_\lambda = \Pi_k \circ (\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ (\Pi_k)^{\otimes k} \circ \Delta_{kk} \circ (\iota_{\lambda_1} \otimes ... \otimes \iota_{\lambda_k}) \circ \Delta_k.$$

Since $\iota_l(x)$ is primitive, we have $\Delta_k(\iota_l(x)) = \iota_l(x) \otimes 1 ... \otimes 1 + 1 \otimes \iota_l(x) ... \otimes 1 + 1 \otimes ... \otimes \iota_l(x)$. Thus we can compute the action of $\Delta_{kk} \circ (\iota_{\lambda_1} \otimes ... \otimes \iota_{\lambda_k})$ on a tensor product $x_1 \otimes ... \otimes x_k \in A^{\otimes k}$ explicitly. The corresponding element in $(A^{\otimes k})^{\otimes k} = (A^{\otimes k}) \otimes ... \otimes (A^{\otimes k})$ is a linear combination of two kinds of tensor products: some of the form $(...) \otimes ... \otimes (1^{\otimes k}) \otimes ... \otimes (...)$ and some which contain one and exactly one term $\iota_{\lambda_i}(x_i)$ in each tensor product $(A^{\otimes k})$. In fact, there are exactly $k!$ such elements, which can be parametrized by the elements of the symmetric group $S_k$. Explicitly, define for $\sigma \in S_k$, the $k^2$-tensor $x_\sigma$ by

$$x_\sigma = y_{11} \otimes y_{21} \otimes ... \otimes y_{k1} \otimes y_{12} \otimes y_{22} \otimes ... \otimes y_{k2} \otimes ... \otimes y_{kk}$$

where $y_{ij} = \iota_{\lambda_i}(x_i)$ if $\sigma(j) = i$, $y_{ij} = 1$ otherwise. Then $\Delta_{kk} \circ (\iota_{\lambda_1} \otimes ... \otimes \iota_{\lambda_k})(x_1 \otimes ... \otimes x_k)$ is equal to $\sum_{\sigma \in S_n} x_\sigma$ plus a sum of tensors $z_{11} \otimes z_{21} \otimes ... \otimes z_{kk}$, where for at least one $j$, one has that all $z_{ij}$ are equal to 1, for $i = 1, ..., k$. If we apply $\Pi_k$ to the whole sum, we obtain $\sum_{\sigma \in S_k} \iota_{\lambda_{\sigma(1)}}(x_{\sigma(1)}) \otimes ... \otimes \iota_{\lambda_{\sigma(k)}}(x_{\sigma(k)})$ plus a sum of tensors having at least a factor 1. Aplying now $\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}$, we obtain $\sum_{\sigma \in S_k} \iota_{\mu_1} \circ \iota_{\lambda_{\sigma(1)}}(x_{\sigma(1)}) \otimes ... \otimes \iota_{\mu_k} \circ \iota_{\lambda_{\sigma(k)}}(x_{\sigma(k)})$, since $\iota_m(1) = 0$. The previous sum vanishes, unless $p(\lambda) = p(\mu)$, which proves (i). In case (ii), since $\iota_m \circ \iota_l = \iota_m$ if $m = l$, and 0 otherwise, we obtain that the sum is equal to $\sum \iota_{\mu_1}(x_{\sigma(1)}) \otimes ... \otimes \iota_{\mu_k}(x_{\sigma(k)})$, where the sum is over all $\sigma$ in $S_k$ such that for $i = 1, ..., k$, $\mu_i = \lambda_{\sigma(i)}$. Note that there are $N$ such permutations. Now,this sum is equal to $\sum(\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \sigma(x_1 \otimes ... \otimes x_k)$, where $\sigma$ denotes here the natural action of the symmetric group on $A^{\otimes n}$. Since the coproduct is cocommutative, we deduce that $(\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ (\Pi_k^{\otimes k}) \circ \Delta_{kk} \circ (\iota_{\lambda_1} \otimes ... \otimes \iota_{\lambda_k}) \circ \Delta_k = \sum(\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \sigma \circ \Delta_k = \sum(\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \Delta_k$, which implies (ii).

□

A similar calculation, which we omit, proves the following result.

**Lemma 3.2** *Let $\mu$,$\lambda$ be two compositions of the same weight such that $l(\mu) > l(\lambda)$. Then:*

$$\iota_\mu \circ \iota_\lambda = 0.$$

**Definition 3.3** *(higher Lie idempotents of the first kind) Let $\lambda$ be a given partition. Then, the element $E_\lambda^\iota$ of $\mathcal{D}_\iota$ is defined by:*

$$E_\lambda^\iota := \frac{1}{l(\lambda)!} \sum_{p(\mu)=\lambda} \iota_\mu.$$

**Remark** If $\iota = e$, the canonical idempotent of $T(X)$, the idempotents $E_\lambda^e$ are the idempotents of [GR]. This follows directly from formulas (9.2.3) and (9.2.4) in [R2], compared to the formula in the previous definition.

**Theorem 3.4** *The $E_\lambda^\iota$ are idempotents. If $\mu$ and $\lambda$ are two partitions such that either $l(\mu) > l(\lambda)$, or $l(\mu) = l(\lambda)$ and $\mu \neq \lambda$, then*

$$E_\mu^\iota \circ E_\lambda^\iota = 0.$$

In general, the $E^\iota_\lambda$ are not orthogonal, since when $l(\mu) < l(\lambda)$, the product does not always vanish.

**Proof**

Let $f = \sum_{p(\alpha)=\lambda} a_\alpha \iota_\alpha$, $a_\alpha \in F$. Then $f \circ f = \sum_{p(\alpha)=p(\beta)=\lambda} a_\alpha a_\beta \iota_\alpha \circ \iota_\beta = \sum_{p(\alpha)=p(\beta)=\lambda} a_\alpha a_\beta N \iota_\alpha$, by Lemma 3.1(ii). Thus $f$ is idempotent if and only if, for any $\alpha$ with $p(\alpha) = \lambda$, one has $\sum_{p(\beta)=\lambda} a_\beta N = 1$.

Now, we take $f = E^\iota_\lambda$. Let $\lambda = 1^{n_1}...k^{n_k}$, and let $S_\lambda$ be the subgroup of the permutations in $S_{l(\lambda)}$ such that: $\forall i \in [1, l(\lambda)], \lambda_i = \lambda_{\sigma(i)}$. Then $S_\lambda \cong S_{n_1} \times ... \times S_{n_k}$ and $N = n_1!...n_k!$. Then, since $n_1 + ... + n_k = l(\lambda)$, $a_\beta = 1/(n_1 + ... + n_k)!$; moreover, the number of $\beta$ with $p(\beta) = \lambda$ is equal to the multinomial coefficient $(n_1 + ... + n_k)!/n_1!...n_k!$. It follows from the previous calculation that $f$ is idempotent.

Now, let $\mu$ and $\lambda$ be as in the theorem. Then $E^\iota_\mu \circ E^\iota_\lambda$ is a linear combination of $\iota_\alpha \circ \iota_\beta$ with $l(\alpha) > l(\beta)$ in the first case, and with $l(\alpha) = l(\beta)$ and $p(\alpha) \neq p(\beta)$ in the second case. In both cases, this product vanishes by Lemma 3.2 and Lemma 3.1. $\square$

# 4 Higher Lie idempotents of the second kind

We define in this section a second family of higher Lie idempotents. They form a complete family of orthogonal idempotents and are the most natural and interesting family of higher Lie idempotents. As a corollary, we give a new and effective proof of the dual Leray theorem.

**Definition 4.1** *(higher Lie idempotents of the second kind). We define elements $F^\iota_\lambda$ in $End(A_n) \subset \mathcal{L}(A)$ by induction on the length of the partition $\lambda$ of weight $n$:*

$$F^\iota_{(n)} := E^\iota_{(n)} = \iota_n,$$

$$F^\iota_\lambda := (1 - \sum_{l(\mu)<l(\lambda)} F^\iota_\mu) \circ E^\iota_\lambda.$$

**Remarks** 1) The $F^\iota_\mu$, which are shown to be othogonal idempotents below, are obtained by an orthogonalisation procedure from the $E^\iota_\mu$; in particular,

if the latter are already orthogonal, one has $E^\iota_\mu = F^\iota_\mu$. In particular, when $A = T(X)$ and $\iota = e$, then the higher Lie idempotents of the second kind are the same as those of [GR].
2) Since, in general, the descent algebra $\mathcal{D}_\iota$ is not stable under composition of morphisms, the $F^\iota_\lambda$ do not belong necessarily to $\mathcal{D}_\iota$. They do, for example, when the Lie idempotents $\iota_n$ belong to $\mathcal{D}(A)$ since, in that case, $\mathcal{D}_\iota = \mathcal{D}(A)$ (this will be proved in Th.5.1).

**Lemma 4.2** *If $l(\mu) > l(\lambda)$, or if $l(\mu) = l(\lambda)$ and $\mu \neq \lambda$, $F^\iota_\mu \circ F^\iota_\lambda = 0$.*

**Proof** By definition, $F^\iota_\lambda$ can be expanded as a linear combination of $E^\iota_{\lambda_1} \circ ... \circ E^\iota_{\lambda_k}$, with $k = 1$ and $\lambda_1 = \lambda$ or $k \geq 2$ and $l(\lambda_1) < l(\lambda)$. Similarly, $F^\iota_\mu$ is of the form $(...) \circ E^\iota_\mu$. Hence Th.3.4 implies the lemma. □

**Theorem 4.3** *The higher Lie idempotents of the second kind form a complete familly of orthogonal idempotents.*

Note that some of these idempotents may be 0, and that "complete" means that the sum of those which act on $A_n$ is equal to $p_n$.
**Proof** First of all, they are idempotents; indeed

$$(F^\iota_\lambda)^2 = F^\iota_\lambda \circ (1 - \sum_{l(\mu)<l(\lambda)} F^\iota_\mu) \circ E^\iota_\lambda = (F^\iota_\lambda - \sum_{l(\mu)<l(\lambda)} F^\iota_\lambda \circ F^\iota_\mu) \circ E^\iota_\lambda;$$

since by Lemma 4.2, $F^\iota_\lambda \circ F^\iota_\mu = 0$ if $l(\mu) < l(\lambda)$ we have:

$$(F^\iota_\lambda)^2 = F^\iota_\lambda \circ E^\iota_\lambda = (1 - \sum_{l(\mu)<l(\lambda)} F^\iota_\mu) \circ (E^\iota_\lambda)^2 = (1 - \sum_{l(\mu)<l(\lambda)} F^\iota_\mu) \circ E^\iota_\lambda = F^\iota_\lambda.$$

We prove that they are orthogonal by induction on the length of the partitions. Let us assume that all the idempotents of the second kind associated to partitions of length less than $k$ are orthogonal. Let $\mu$ and $\lambda$ be distinct partitions, of length $\leq k$, with one of them of length $k$. By Lemma 4.2, all we have to show is that $F^\iota_\mu \circ F^\iota_\lambda = 0$ when $l(\mu) < k$ and $l(\lambda) = k$. Then:

$$F^\iota_\mu \circ F^\iota_\lambda = F^\iota_\mu \circ (1 - \sum_{l(\beta)<l(\lambda)} F^\iota_\beta) \circ E^\iota_\lambda;$$

but by the induction hypothesis: $F^\iota_\mu \circ F^\iota_\beta = \delta_{\mu\beta}$, so that:

$$F^\iota_\mu \circ F^\iota_\lambda = F^\iota_\mu \circ (1 - F^\iota_\mu) \circ E^\iota_\lambda = 0,$$

and the proof of the orthogonality is complete.

Finally, since $(F^\iota_\mu)_{|\mu|=n}$ is a familly of orthogonal idempotents, to prove that it is a complete familly amounts to prove that $\sum\limits_{|\mu|=n} Im(F^\iota_\mu) = A_n$ (this sum is actually direct by orthogonality). But, by the triangularity condition in the definition of the $F^\iota_\mu$'s, this amounts to prove that $\sum\limits_{|\mu|=n} Im(E^\iota_\mu) = A_n$: indeed, we have by Def.4.1 that $E^\iota_\lambda(x) = F^\iota_\lambda(x)$ plus a sum of $E^\iota_{\lambda_1} \circ ... \circ E^\iota_{\lambda_k}$ with $l(\lambda_1) < l(\lambda)$, which proves inductively that the equality with the images of the $E$'s implies the one with the $F$'s.

Since $A$ is a graded cocommutative connected bialgebra of characteristic zero, it is by the Cartier-Milnor-Moore theorem isomorphic to the enveloping algebra of $Prim(A)$. Hence, by the Poincaré -Birkhoff-Witt theorem it is the direct sum of its subspaces $A^\lambda$, where for any partition $\lambda$, the latter subspace is spanned by the elements $(a_1, ..., a_k) = (1/k!) \sum_{k\in S_k} a_{\sigma(1)}...a_{\sigma(k)}$, for any choice of homogeneous primitive elements $a_i$, with $deg(a_i) = \lambda_i$ and $\lambda = (\lambda_1, ..., \lambda_k)$. Now, let us verify that the restriction of $E^\iota_\lambda$ to $A^\lambda$ is the identity, which will finish the proof.

Indeed, if the $a_i$ are as above, consider the sum $\sum_\mu \iota_{\mu_1} * ... * \iota_{\mu_k}(a_1...a_k)$, over all compositions which are a rearrangement of the partition $\lambda$. It is equal to $\sum_\mu \Pi_k \circ \iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k} \circ \Delta_k(a_1...a_k)$, and this, since the $a_i$ are primitive and since $\iota_m$ is the identity on the homogeneous primitive elements of degree $m$, is equal to $\sum_{\sigma\in S_k} a_{\sigma(1)}...a_{\sigma(k)}$. Since the same result holds if we replace the product $a_1...a_k$ by one of its permutation, and since $E^\iota_\lambda$ is equal by definition to $(1/k!) \sum_\mu \iota_\mu$, we conclude that $E^\iota_\lambda(a_1, ..., a_k)$ is equal to $(a_1, ..., a_k)$, which shows that $E^\iota_\lambda$ is the identity on $A^\lambda$. □

Recall that the *cofree cocommutative coalgebra* on a vector space $V$ over $F$ is the space $\bigoplus\limits_{n\in\mathbf{N}} (V^{\otimes n})^{S_n}$, the direct sum of the $S_n$-invariants in the tensor algebra over $V$, with the coproduct inherited from this tensor algebra (that is, the one for which elements of $V$ are primitive and which is a homomorphism of the tensor algebra).

**Corollary 4.4** *(effective dual Leray Theorem) The canonical map:*

$$\bigoplus_{n\in\mathbf{N}} \iota^{\otimes n}\circ\Delta_n : \ A \quad \mapsto \quad \bigoplus_{n\in\mathbf{N}}((Prim\ A)^{\otimes n})^{S_n}$$

*is a coalgebra isomorphism between $A$ and the cofree cocommutative coalgebra over $Prim\ A$. Moreover, denote by $Sym^\lambda(Prim(A))$ the subspace of $(Prim(A)^{\otimes k})^{S_k}$ spanned by the elements $(1/k!)\sum_{\sigma\in S_k} x_{\sigma(1)}\otimes ...\otimes x_{\sigma(k)}$, with $x_i \in Prim(A)$ homogeneous, $deg(x_i)=\lambda_i$, $\lambda=(\lambda_1,...,\lambda_k)$. Then the restriction of the inverse isomorphism to $Sym^\lambda(Prim(A))$ is the map: $\frac{1}{l(\lambda)!}(1-\sum\limits_{l(\mu)<l(\lambda)} F^\iota_\mu)\circ\Pi_k$.*

The corollary follows, once it is noted that $Sym^\lambda(Prim(A))$ is canonically isomorphic to $A^\lambda$, through the map $\Pi_k$.

# 5 Higher Lie idempotents of the third kind

Using the formal algebraic properties of noncommutative symmetric functions, Krob, Leclerc and Thibon[KLT] have shown that, given a Lie idempotent $\iota$ (in the tensor algebra on an infinite alphabet) whose components $\iota_n$ belong to the descent algebra, a family of higher Lie idempotents, which generalize the idempotents of [GR], can be associated to it. These idempotents have interesting combinatorial properties. Our approach makes the link with higher Lie idempotents of the second kind; it sheds some light on the combinatorial meaning of these idempotents.

In the next result, we give necessary and sufficient conditions for the equality of the descent algebra $\mathcal{D}(A)$ of $A$ and the $\iota$-descent algebra $\mathcal{D}_\iota$.

**Theorem 5.1** *Let $\iota$ be a Lie idempotent. The following conditions are equivalent:*
*(i) $\mathcal{D}(A)=\mathcal{D}_\iota$;*
*(ii) For any $n$, $\iota_n\in\mathcal{D}(A)$.*
*(iii) For any $n$, $p_n\in\mathcal{D}_\iota$.*
*(iv) $\mathcal{D}_\iota$ is closed under composition.*

**Proof** That (i) implies (ii) is clear; suppose that (ii) holds. Denote by $e$ the *canonical idempotent*, i.e. the one defined by the formula $e = log_*(id)$, where $log_*$ is the logarithm in the convolution algebra. Then $\mathcal{D}(A) = \mathcal{D}_e$, so that $\iota_n$ can be expanded with respect to the $e_\lambda$. We thus have (*) $\iota_n = \alpha e_n + \sum_\mu * e_\mu$, where the sum is over the composition $\mu$ of weight $n$ and not equal to $(n)$, and where $\alpha$ and $*$ denote scalars. We multiply this by $e_n$ on the right in $\mathcal{L}(A)$; since $\iota$ is a Lie idempotent, we have $e_n = \iota_n \circ e_n$. Thus we obtain $e_n = \alpha e_n$, since $e_\mu \circ e_n = 0$ by Lemma 3.2. Thus, in case $e_n \neq 0$, $\alpha = 1$; and in case $e_n = 0$, we must have also $\iota_n = 0$, and we may take $\alpha = 1$ in (*). This shows that for a suitable ordering, we may write down a unipotent transition matrix fom the $e_\lambda$'s to the $\iota_\lambda$'s. Hence they span the same subspace and $\mathcal{D}(A) = \mathcal{D}_\iota$.

Now, that (i) implies (iii) is also clear. Suppose that (iii) holds. Then, since $\mathcal{D}_\iota$ is closed under convolution, it contains $\mathcal{D}(A)$, hence the $e_n$. We then argue as above (exchanging $\iota$ and $e$) to conclude that $\mathcal{D}_\iota = \mathcal{D}_e = \mathcal{D}(A)$.

It is clear that (i) implies (iv). Now, if (iv) holds then, since $\mathcal{D}_\iota$ is closed under convolution, it contains the higher Lie idempotents of the first kind by Def.3.3, hence also the higher Lie idempotents of the second kind by Def.4.1. But we have by Th.4.3

$$\sum_{|\mu|=n} F^\iota_\mu = p_n.$$

It follows that the $p_n$ are in $\mathcal{D}_\iota$, thus (iii) holds, hence also (i). □

In the sequel we assume that $\mathcal{D}(A) = \mathcal{D}_\iota$. Then we may write, for some scalars $a^\iota_\mu$,

$$p_n = \sum_{|\mu|=n} a^\iota_\mu \iota_\mu.$$

The following definition generalizes [GR] and [KLT].

**Definition 5.2** *(higher Lie idempotents of the third kind) For any partition $\lambda$, we set:*

$$\mathcal{E}^\iota_\lambda := \sum_{p(\mu)=\lambda} a^\iota_\mu \cdot \iota_\mu.$$

**Remark** By Def.3.3 and the remark in Section 3, the $\mathcal{E}^\iota_\lambda$ reduce to the idempotents of [GR] if one takes $\iota = e$ in $A = T(X)$.

**Lemma 5.3** *The higher idempotents of the third kind are quasi-idempotents.*

Recall that a *quasi-idempotent* is an element whose square is a scalar multiple of itself.

**Proof** In fact, according to Lemma 3.1(ii), any linear combination of the $\iota_\mu$s with $p(\mu) = \lambda$, for a given partition $\lambda$, is quasi-idempotent. Indeed, for any familly $(b_\mu)_{p(\mu)=\lambda}$ of coefficients

$$(\sum_{p(\mu)=\lambda} b_\mu \iota_\mu)^2 = \sum_{p(\mu_1)=p(\mu_2)=\lambda} b_{\mu_1} b_{\mu_2} \iota_{\mu_1} \circ \iota_{\mu 2}$$

$$= \sum_{p(\mu_1)=p(\mu_2)=\lambda} b_{\mu_1} b_{\mu_2} N \iota_{\mu_1} = (\sum_{p(\mu)=\lambda} b_\mu) N (\sum_{p(\mu)=\lambda} b_\mu \iota_\mu).$$

□

**Theorem 5.4** *The higher Lie idempotents of the third kind form a complete family of orthogonal idempotents.*

**Proof** Let us choose a total order $>$ of the set of partitions $\lambda$ of a given weight $n$, such that $\lambda > \nu$ if $l(\lambda) > l(\nu)$: we order the partitions first by length, then arbitrarily in the set of partitions of a given length. The proof of the theorem is by decreasing induction relatively to this ordering.

Observe first that, according to lemmas 3.1 and 3.2:

$$\mathcal{E}^\iota_\lambda \circ \mathcal{E}^\iota_\mu = 0 \quad if \quad \lambda > \mu.$$

In particular, if we set $[n] := (1, ..., 1)$ (the partition of $n$ of maximal length), then:

$$\mathcal{E}^\iota_{[n]} \circ \mathcal{E}^\iota_\mu = 0$$

for any partition $\mu \neq [n]$. In particular, $\mathcal{E}^\iota_{[n]} \circ (\sum_{\mu<[n]} \mathcal{E}^\iota_\mu) = 0$ and:

$$\mathcal{E}^\iota_{[n]} = \mathcal{E}^\iota_{[n]} \circ p_n = \mathcal{E}^\iota_{[n]} \circ (\sum_{\mu \leq [n]} \mathcal{E}^\iota_\mu) = (\mathcal{E}^\iota_{[n]})^2.$$

Moreover:

$$(\sum_{\mu<[n]} \mathcal{E}^\iota_\mu)^2 = (1 - \mathcal{E}^\iota_{[n]})^2 = 1 - \mathcal{E}^\iota_{[n]} = \sum_{\mu<[n]} \mathcal{E}^\iota_\mu,$$

and:

$$(\sum_{\mu<[n]} \mathcal{E}^\iota_\mu) \circ \mathcal{E}^\iota_{[n]} = (1 - \mathcal{E}^\iota_{[n]}) \circ \mathcal{E}^\iota_{[n]} = 0.$$

In other words, $\mathcal{E}^\iota_{[n]}$ and $\sum_{\mu<[n]} \mathcal{E}^\iota_\mu$ are two orthogonal idempotents.

Let us assume by induction that, for a given $\lambda$, $(\mathcal{E}^\iota_\beta)_{\beta>\lambda} \cup \{f\}$ is a complete family of orthogonal idempotents, where:

$$f := \sum_{\mu\leq\lambda} \mathcal{E}^\iota_\mu.$$

We set: $g := \sum_{\mu>\lambda} \mathcal{E}^\iota_\mu$, $k := \sum_{\mu<\lambda} \mathcal{E}^\iota_\mu$, $h := \mathcal{E}^\iota_\lambda$, $b^\iota_\lambda := \sum_{p(\mu)=\lambda} a^\iota_\mu \cdot N(\lambda)$ and $b := b^\iota_\lambda$. where $N(\lambda)$ is defined in Lemma 3.1.(ii).

We are going to prove that $(k, h, g)$ is a complete family of orthogonal idempotents. This implies that $(\mathcal{E}^\iota_\beta)_{\beta\geq\lambda} \cup \{k\}$ is also a complete family of orthogonal idempotents, hence the proof of the theorem by induction.

If $h = 0$, the property is clear, so that we can assume $h \neq 0$ in what follows.

First of all, since $\mathcal{E}^\iota_\beta \circ \mathcal{E}^\iota_\gamma = 0 \;\; if \;\; \beta > \gamma$, we have for any $\mu \leq \lambda$ and any $\beta < \lambda$:

$$g \circ \mathcal{E}^\iota_\mu = 0 \textit{ and } h \circ \mathcal{E}^\iota_\beta = 0.$$

In particular:

$$g \circ k = g \circ f = g \circ h = 0$$

and

$$h \circ k = 0.$$

**Claim 5.5** *We have:* $b = 1$.

Indeed:

$$h \circ 1 = h \circ (h + g + k) = bh + h \circ g,$$

since $h \circ h = bh$ (see the proof of Lemma 5.3). Thus

$$h \circ g = (1 - b)h.$$

On the other hand, according to the induction hypothesis, we have:

$$(h+k)\circ g = f\circ g = 0,$$

since $h+k=f$, so that:

$$k\circ g = (b-1)h.$$

By left composition with $h$, we get: $0 = h\circ k\circ g = b(b-1)h$. Since $h\neq 0$, $b$ is equal to 0 or 1.

To show that $b$ is equal to 1, we interpolate between $\iota$ and the canonical Lie idempotent $e$ (see the example in Section 2) and use a continuity argument to conclude. The canonical Lie idempotents $e_n$ belong to the descent algebra $\mathcal{D}(A)$. The associated higher idempotents of the third kind are the idempotents of [GR]. By the very definition of these idempotents, $b_\lambda^e = 1$ for any partition $\lambda$. Let us introduce the one parameter family of Lie idempotents:

$$\iota_n^\epsilon := \epsilon\cdot\iota_n + (1-\epsilon)\cdot e_n,\ \ \epsilon\in[0,1]\cap\mathbf{Q}.$$

We have: $\iota_n^0 = e_n$ and $\iota_n^1 = \iota_n$. To any $\epsilon\in[0,1]\cap\mathbf{Q}$ are associated higher Lie idempotents of the third kind $(\mathcal{E}_\mu^{\iota^\epsilon})_{|\mu|=n}$, which can be computed by using their definition, the definition of the higher idempotents of the second kind and the decomposition:

$$p_n = \sum_{|\mu|=n} F_\mu^{\iota^\epsilon}.$$

It follows that the coefficients $a_\mu^{\iota^\epsilon}$ of the higher Lie idempotents of the third kind depend polynomially of $\epsilon$. In particular, for any partition $\lambda$, the coefficient $b_\lambda^{i^\epsilon}$ depends polynomially of $\epsilon$. Since the polynomial function $\epsilon\longmapsto b_\lambda^{\iota^\epsilon}$ has its values in the discrete set $\{0,1\}$ and since $b_\lambda^{\iota^0} = b_\lambda^e = 1$, we have $b_\lambda^{\iota^\epsilon} = 1$ for all $\epsilon\in[0,1]\cap\mathbf{Q}$ and the proof of the claim is complete: $b = b_\lambda^\iota = 1$.

Let us return to the theorem. The following identities hold:

$$g^2 = g,\ \ g\circ h = 0,\ \ g\circ k = 0,\ \ h^2 = h,\ \ h\circ k = 0,\ \ h\circ g = 0, k\circ g = 0.$$

We still have to prove that: $k^2 = k$ and $k\circ h = 0$. Expanding the identities $p_n\circ k = k$ and $k\circ p_n = k$, we get:

$$k = p_n\circ k = (g+k+h)\circ k = g\circ k + k\circ k + h\circ k = k^2;$$

$$k = k \circ p_n = k \circ (g + k + h) = k \circ g + k \circ k + k \circ h = k + k \circ h,$$

and the proof of theorem 5.3 is complete. □

**Remark** Note that the higher Lie idempotents of the second kind may be written

$$F_\mu^\iota = \sum_{|\nu|=n} \alpha_\nu^\mu \cdot \iota_\nu,$$

and that the coefficients may be computed using Def.3.3 and Def.4.1. Since $\sum_{|\mu|=n} F_\mu^\iota = p_n$, we obtain, assuming that the $\iota_\mu$ are linearly independant (for example when $A = T(X)$),

$$a_\mu^\iota = \sum_{|\nu|=n} \alpha_\mu^\nu.$$

Hence our approach provides a simple algorithmic computation of the coefficients $a_\mu^\iota$ (compare [KLT]).

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