Regularity from two perspectives
Comprehensible Seminar
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Given an infinite sequence it is natural to wish to determine how “complex” it is. In combinatorics on words, some complexity of sequences (for example, the 2-abelian complexity of the Thue–Morse sequence) are regular. Such complexity sequences look very similar to functions that are continuous on $\mathbb{R}$ but nowhere differentiable. Classical examples of these functions are Brownian motion and Weierstraß functions. Given two continuous functions which are nowhere differentiable, it is natural to compare them and try to determine which one is, at a given point, more regular than the other one. The local regularity of a function at a given point can be characterized via its Hölder exponent.

In this talk, we present numerical experiments that exhibit sequences that are regular from the combinatorics point of view and also from the analysis point of view.

Main definitions

- In combinatorics on words, the $k$-kernel of a sequence $(x_n)_{n \geq 0}$ is the set

$$\mathcal{K} = \{ (x_{kn+r})_{n \geq 0} \mid e \geq 0, 0 \leq r \leq k^e \}$$

composed of particular subsequences of $(x_n)_{n \geq 0}$. A sequence $(x_n)_{n \geq 0}$ is called $k$-regular if the $\mathbb{Z}$-module generated by its $k$-kernel is finitely generated.

- In analysis, a locally bounded function $f : \mathbb{R} \to \mathbb{R}$ belongs to the Hölder space $C^s(t)$, with $t \in \mathbb{R}$ and $s \geq 0$, if there exist a constant $C$ and a polynomial $P$ with $\deg(P) < s$ such that

$$|f(t + \ell) - P(\ell)| \leq C|\ell|^s$$

in a neighborhood of 0. The Hölder exponent of $f$ at $t$ is defined by

$$h_f(t) = \sup\{s \geq 0 : f \in C^s(t)\}.$$