ON THE COHERENCE CONDITIONS FOR PSEUDO-DISTRIBUTIVE LAWS

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Abstract. Pseudo-distributive laws encode the fundamental properties necessary to combine different forms of algebraic structure on a category. We extend and refine the existing results on pseudo-distributive laws. In our approach, results concerning pseudo-distributive laws are derived from more general results concerning lax distributive laws. This approach allows us to clarify the formulation of the coherence conditions for pseudo-distributive laws.

1. Introduction

One of the fundamental properties of the presheaf construction is that the category of presheaves over a monoidal category can be equipped with a canonical monoidal structure [8]. This monoidal structure, generally known as the convolution tensor product, has many important applications in different areas of mathematics. For example, it is used in the construction of the category of symmetric spectra [13], provides the operation of product for combinatorial species of structures [15], and allows us to describe operads as monoids [18]. Many of these applications rely on the fact that categories of presheaves over a monoidal category are not only monoidal and cocomplete, but also monoidally cocomplete, in the sense that the tensor product preserves colimits in both arguments [14]. The notion of a pseudo-distributive law captures the fundamental properties of this form of interaction between different forms of algebraic structure on a category. Pseudo-distributive laws generalise strict distributive laws in order to allow us to describe correctly the many examples in which different forms of algebraic structure interact via isomorphisms rather than equalities [16]. For example, in a category with products and coproducts, the former distribute over the latter only up to isomorphism.

Our aim here is to extend and refine the existing results concerning pseudo-distributive laws [7, 16, 25, 29]. The general goal of the research in this area is to extend the results for distributive laws in the theory of monads [1, 23], to the context of two-dimensional monad theory [4]. Jon Beck’s fundamental theorem concerning distributive laws states that there is a bijection between distributive laws of a monad over another monad and liftings of the second monad to the category of Eilenberg-Moore algebras for the first [2, 1]. A preliminary issue in two-dimensional monad theory is that, alongside the notion of a strict algebra, there are the more general notions of a pseudo-algebra and a lax algebra [28]. The notion of a lax algebra generalises the notion of a strict algebra by replacing the commutative diagrams for the latter notion with diagrams filled with a specified 2-cell. When this 2-cell is invertible, we have a pseudo-algebra. Pseudo-algebras can therefore be seen both as special lax algebras or as strict algebras ‘up to isomorphism’. For example, strict monoidal categories, unbiased monoidal categories, and lax unbiased monoidal categories are respectively the strict algebra, pseudo-algebras,

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and lax algebras for a 2-monad on the 2-category of small categories, functors, and natural transformations [21]. These distinctions can be replicated in the case of morphisms, so as to introduce strict maps, pseudo-maps, and lax maps. In generalising the correspondence between distributive laws and liftings, it is therefore necessary to decide on which kind of notion to focus the attention. For many applications, pseudo-algebras and pseudo-maps are focus of the attention, and in some cases pseudo-algebras can be replaced by strict algebras via general forms of coherence theorems [4, 16, 26]. By their very definition, however, pseudo-algebras and pseudo-maps are special forms of the corresponding lax concepts. It is therefore natural to work with lax algebras, and then specialize the results to the case of pseudo-algebras. This approach opens the possibility of new applications in which genuine lax algebras are involved. We will therefore introduce the notion of a lax distributive law and regard pseudo-distributive laws as the special lax distributive laws. As in the formulation of the notions of lax algebra, in a lax distributive law, the four diagrams expressing the axioms for a distributive law are replaced by diagrams containing specified 2-cells, that are required to satisfy appropriate coherence conditions. When these modifications are invertible, we have a pseudo-distributive law. Another reason to consider pseudo-concepts as special cases of lax concepts rather than in isolation is to keep track of the directions of 2-cells. Indeed, in many examples these 2-cells arise as canonical natural transformations with a specified direction, which can be then shown to be invertible [11].

Our main result states that lax distributive laws and liftings to 2-categories of lax algebras are related by a 2-adjunction. This result may come as a surprise to the readers familiar with the existing literature. An inspection of the proofs in [25, 29] reveals that the invertibility of the modifications that are part of the data for a pseudo-distributive law is used extensively to relate pseudo-distributive laws and liftings to 2-categories of pseudo-algebras, thus giving the impression that an analogous result is not available for the lax case. This 2-adjunction then restricts to a 2-equivalence between pseudo-distributive laws and liftings to 2-categories of pseudo-algebras, a theorem originally obtained by Miki Tanaka in [29] by extending the results of Francisco Marmolejo in [25].

The definition of a lax distributive law for pseudo-monads will be formulated here as the natural result of the study of the notions of lax pseudo-monad map, lax pseudo-monad transformation, and pseudo-monad modification. This is suggested by the illuminating analysis of distributive laws provided by the formal theory of monads [27] and by the methods used in [29]. This approach is extremely useful to guide the formulation of the coherence conditions for lax distributive laws. We formulate ten coherence axioms for a lax distributive law, extending the nine coherence axioms for pseudo-distributive laws considered in [25]. A version of the tenth coherence condition was originally introduced for pseudo-distributive laws in [29], where it was considered as missing from the axiomatisation in [25]. Although it is a crucial axiom to consider in the context of lax distributive laws, we show that it is not necessary to obtain the theorem relating pseudo-distributive laws and liftings to 2-categories of pseudo-algebras stated in [29]. The clarification of the formulation of the coherence axioms for a pseudo-distributive law was indeed one of the original motivations for the research presented here. For the convenience of the readers, we will also present these coherence conditions in a geometrically intuitive way, following [19, 28].

We work here with pseudo-monads rather than with 2-monads. Many examples of pseudo-distributive involve the presheaf construction which, apart from size issues on which we return at the end of the paper, is a pseudo-monad rather than as a 2-monad. A generalisation of the formal theory of monads has been studied
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in [20, 25] by considering pseudo-monads defined within a Gray-enriched category. Here Gray is the category of 2-categories and 2-functors equipped with the so-called Gray tensor product [12]. Since Gray is enriched in itself, it can be regarded as the tricategory of 2-categories, 2-functors pseudo-natural transformations, and modifications [12]. The general theory developed in [20, 25] leads therefore to results concerning pseudo-monads in Gray itself. As observed in [7], this approach runs into the problem that the Kleisli construction for a pseudo-monad on a 2-category, even when the underlying pseudo-functor is a 2-functor, produces a bicategory rather than a 2-category, and thus leads outside Gray. A remedy to this issue would be to redevelop the formal theory of pseudo-monads within a general tricategory [12]. This would then give as a special case results concerning the tricategory Bicat of bicategories, pseudo-functors, pseudo-natural transformations, and modifications. Our approach here is to work explicitly with pseudo-monads defined within the sub-tricategory of Bicat whose 0-cells are 2-categories. This approach is general enough to capture the examples of interest and leaves open the possibility to obtain a theorem relating pseudo-distributive laws and extensions of pseudo-monads to Kleisli bicategories.

Notation. We assume familiarity with the notions of 2-category, pseudo-functor, and pseudo-natural transformation [3, 19]. By coherence results [12, 26] we can omit mention of pseudo-functoriality, and treat pseudo-functors as if they were 2-functors. As in [12, Section 2.1] we assume that the 2-cells associated to a pseudo-natural transformation \( p : H \to K \) between pseudo-functors have the following direction:

\[
\begin{array}{c}
HA \xrightarrow{Hf} HB \\
pA \downarrow \quad \downarrow p_f \\
KA \xrightarrow{Kf} KB
\end{array}
\]

This choice of direction isparangeparate from the one described in [3].

Convention. For brevity, we often avoid defining explicitly the ‘pseudo’ version of the lax concepts that we introduce. We assume implicitly that they are defined by requiring the invertibility of the appropriate 2-cells.

2. PSEUDO-MONADS

Let us recall that a pseudo-monad \( S \) on a 2-category \( X \) consists of a pseudo-functor \( S : X \to X \), two pseudo-natural transformations \( u : 1d_X \to S \) and \( m : S^2 \to S \), called unit and multiplication, and invertible modifications \( \alpha \), \( \lambda \), and \( \rho \) fitting in the diagrams in (1) and satisfying the coherence conditions (PSM1), (PSM2) given in Appendix A.

\[
\begin{array}{c}
S^3 \xrightarrow{Sm} S^2 \\
mS \downarrow \quad \downarrow \alpha \\
S^2 \xrightarrow{m} S
\end{array}
\]

We refer to the modifications \( \alpha \), \( \lambda \) and \( \rho \) as the associativity, left unit, and right unit of the pseudo-monad, respectively. Note that, in contrast with the existing literature on pseudo-monads [7, 20, 25], we chose the canonical directions for these modifications as in the definition of the more general notion of lax monad [5]. This choice of directions agrees with the notion of a lax algebra for a 2-monad [28, 16],
in the sense that the free pseudo-algebras determined by the pseudo-monad can be seen as lax algebras without inverting any 2-cell. In [24] it has been shown that (PSM1) and (PSM2) imply three further coherence conditions, labelled (PSM3), (PSM4), (PSM5) in Appendix A. Furthermore, they suffice to deduce a coherence theorem for pseudo-monads [20]. In the following, we refer to a pair of the form \((X, S)\) where \(X\) is a 2-category and \(S\) is a pseudo-monad on it simply as a pseudo-monad.

We introduce the notions of lax pseudo-monad map, lax pseudo-monad transformation, and pseudo-monad modifications. As we will see in Section 5, the coherence conditions for these notions determine the coherence conditions for lax distributive laws. A lax pseudo-monad map \((H, h) : (X, S) \to (Y, T)\) between pseudo-monads \((X, S)\) and \((Y, T)\) consists of a pseudo-functor \(H : X \to Y\), a pseudo-natural transformation \(h : TH \to HS\), and two modifications, fitting in the diagrams in (2) below and subject to the three coherence conditions (M1), (M2), and (M3) given in Appendix A. Observe that the notion of lax 2-monad map defined in [16, Section 4] is the special case of the notion defined above that arises by considering 2-monads \(S\) and \(T\) on the same 2-category, with the identity pseudo-functor as \(H\).

\[
\begin{array}{cccc}
T^2 H & \xrightarrow{T H} & TH S & H \xrightarrow{\nu H} TH \\
\downarrow \alpha H & & \downarrow h S & \downarrow h \\
TH & \xrightarrow{h} & HS & \end{array}
\]

A lax pseudo-monad transformation \((p, \bar{p}) : (H, h) \to (K, k)\) between lax pseudo-monad maps consists of a pseudo-natural transformation \(p : H \to K\) and of a modification \(\bar{p}\) fitting in the diagram (3), and subject to the two coherence conditions (T1) and (T2) presented in Appendix A.

\[
\begin{array}{cccc}
TH & \xrightarrow{T p} & TK & \\
\downarrow h & & \downarrow k & \\
HS & \xrightarrow{p S} & KS & \end{array}
\]

Finally, a pseudo-monad modification \(\alpha : (p, \bar{p}) \to (q, \bar{q})\) between pseudo-monad transformations is a modification \(\alpha : p \to q\) of pseudo-natural transformations which satisfies the single coherence condition (MD) given in Appendix A. There is then an obvious 2-category \(\text{Psm}_{\text{lax}}([X, S], (Y, T)]\) of lax pseudo-monad maps, lax pseudo-monad transformations, and pseudo-monad modifications. We write \(\text{Psm}(X, S), (Y, T]])\) for the sub-2-category of pseudo-maps, pseudo-transformations, and modifications.

3. Liftings

**Algebras.** Given a pseudo-monad \((X, S)\), we write \(\text{Lax-S-Alg}\) for the 2-category of lax algebras, lax algebra maps, and algebra 2-cells. Since these notions will be used extensively in the following, we recall their definition. This also allows us to fix some notation. A lax algebra consists of a 0-cell \(A\) in \(X\), called the underlying 0-cell of the algebra, a 1-cell \(a : SA \to A\), called the structure map of the algebra,
and 2-cells $\bar{a}$ and $\tilde{a}$, generally referred to as the associativity and unit 2-cells of the algebra, fitting in the diagrams below:

and subject to the following three coherence conditions:

As usual, we refer to a lax algebra simply by the name of its underlying 0-cell, unless it is necessary to do otherwise to avoid confusion. An analogous convention is assumed for other structures. We have a pseudo-algebra when the associativity and unit 2-cells of the lax algebra are invertible. As shown in [24], the third coherence condition involved in the definition of a lax algebra is redundant for pseudo-algebras. Given lax algebras $A$ and $B$, a lax map between them consists of a 1-cell $f : A \to B$. 
and a 2-cell fitting in the following diagram

\[
\begin{array}{c}
SA \xrightarrow{sf} SB \\
\downarrow a \quad \downarrow f \\
A \xrightarrow{f} B
\end{array}
\]

subject to the following two coherence conditions:

\[
\begin{array}{c}
\begin{array}{c}
S^2 A \xrightarrow{s^2 f} S^2 B \\
\downarrow m_A \quad \downarrow Sf \quad \downarrow Sb \\
SA \xrightarrow{a} SB \\
\downarrow a \quad \downarrow f \\
A \xrightarrow{f} B
\end{array} = \\
\begin{array}{c}
S^2 A \xrightarrow{s^2 f} SB \\
\downarrow m_A \quad \downarrow Sf \quad \downarrow Sb \\
\downarrow b \\
A \xrightarrow{f} B
\end{array}
\end{array}
\]

When \( \bar{f} \) is invertible, we say that \( f : A \to B \) is a pseudo-map. Finally, an algebra 2-cell \( \alpha : (f, \bar{f}) \to (g, \bar{g}) \) is a 2-cell \( \alpha : f \to g \) subject to a single coherence axiom

\[
\begin{array}{c}
\begin{array}{c}
S^3 A \xrightarrow{s^3 f} S^3 B \\
\downarrow Sf \quad \downarrow Sg \\
\downarrow \alpha_A \\
\downarrow Sf \quad \downarrow Sg \\
SA \xrightarrow{a} SB \\
\downarrow a \quad \downarrow g \\
A \xrightarrow{g} B
\end{array} = \\
\begin{array}{c}
S^3 A \xrightarrow{s^3 f} SB \\
\downarrow Sf \quad \downarrow Sg \\
\downarrow \lambda_A \\
\downarrow Sf \quad \downarrow Sg \\
\downarrow b \\
A \xrightarrow{g} B
\end{array}
\end{array}
\]

Note that there is an obvious forgetful 2-functor \( \text{Lax-S-Alg} \to X \). In [5] it is shown that this 2-functor has a left lax adjoint \( X \to \text{Lax-S-Alg} \). A 0-cell \( A \) in \( X \) is mapped by the left lax adjoint into the free pseudo-algebra \( SA \) on \( A \), with structure map \( m_A : S^2 A \to SA \) and associativity and unit 2-cells given as follows:

\[
\begin{array}{c}
\begin{array}{c}
S^3 A \xrightarrow{s^3 A} S^2 A \\
\downarrow m_A \\
S^2 A \xrightarrow{m_A} SA \\
\downarrow \alpha_A \\
\downarrow m_A \\
S^2 A \xrightarrow{1_A} SA \\
\downarrow m_A
\end{array} = \\
\begin{array}{c}
SA \\
\downarrow u_{SA} \\
S^2 A \\
\downarrow m_A \\
SA
\end{array}
\end{array}
\]
This lax adjunction restricts to a pseudo-adjunction when Lax-S-Alg is replaced by Ps-S-Alg, the 2-category of pseudo-algebras, pseudo-maps, and algebra 2-cells [7].

**Liftings.** The notions of lax pseudo-monad map, lax pseudo-monad transformation, and pseudo-monad modification introduced in Section 2 describe the structure on pseudo-functors, pseudo-natural transformations, and modifications that allows us to lift them to the level of lax algebras. In order to make this precise, we define the notions of a lifting for pseudo-functors, pseudo-natural transformations, and modifications. For pseudo-monads \((X, S)\) and \((Y, T)\), a lifting of a pseudo-functor \(H : X \to Y\) consists of a pseudo-functor \(\hat{H} : \text{Lax-S-Alg} \to \text{Lax-T-Alg}\) that makes the following diagram commute:

\[
\begin{array}{ccc}
\text{Lax-S-Alg} & \xrightarrow{H} & \text{Lax-T-Alg} \\
U & \downarrow & U \\
X & \xrightarrow{H} & Y \\
\end{array}
\]

The notions of lifting of a pseudo-natural transformation and of a modification can be defined in an analogous way. For liftings \((H, \hat{H})\) and \((K, \hat{K})\), a lifting of a pseudo-natural transformations \(p : H \to K\), consists of a pseudo-natural transformation \(\hat{p} : \hat{H} \to \hat{K}\) making the following diagram commute:

\[
\begin{array}{ccc}
U \hat{H} & \xrightarrow{U\hat{p}} & U \hat{K} \\
\downarrow^{id} & & \downarrow^{id} \\
H U & \xrightarrow{pU} & K U \\
\end{array}
\]

Note that the vertical arrows in the diagram above are identities by the definition of lifting of a pseudo-functor. Finally, given two liftings \(\hat{p} : \hat{H} \to \hat{K}\), \(\hat{q} : \hat{H} \to \hat{K}\) of pseudo-natural transformations \(p : H \to K\), \(q : H \to K\), a lifting of a modification \(\alpha : p \to q\) is a modification \(\hat{\alpha} : \hat{p} \to \hat{q}\) making the diagram below commute:

\[
\begin{array}{ccc}
U \hat{p} & \xrightarrow{U\hat{\alpha}} & U \hat{q} \\
\downarrow^{id} & & \downarrow^{id} \\
\hat{p} U & \xrightarrow{\alpha U} & \hat{q} U \\
\end{array}
\]

Again, the vertical arrows above are identities by the definition of lifting of a pseudo-natural transformation.

**Construction of liftings.** We define \(\text{Lift}_{\text{Lax}}[(X, S), (Y, T)]\) be the 2-category of liftings of pseudo-functors, pseudo-natural transformations, and modifications. We define a 2-functor

\[
\begin{aligned}
\text{Psm}_{\text{Lax}}[(X, S), (Y, T)] & \longrightarrow \quad \text{Lift}_{\text{Lax}}[(X, S), (Y, T)] \\
F & \quad F
\end{aligned}
\]

This provides a first indication that the coherence conditions for pseudo-monad maps, transformations, and modifications have been formulated correctly. Given a lax pseudo-monad map \((H, h) : (X, S) \to (Y, T)\), with modifications as in (2), we wish to define a lifting of \(H : X \to Y\). Let \(A\) be a lax \(S\)-algebra. We define a lax \(T\)-algebra \(HA\) with structure map given by the composite

\[
THA \xrightarrow{h_A} HSA \xrightarrow{H_a} HA
\]
The associativity and unit 2-cells of the lax algebra are provided by the pasting diagrams below, in which we make use the modifications in (2) and the lax algebra structure on $A$.

### Lemma 3.1.

If $A$ is a lax $S$-algebra, then $HA$ is a lax $T$-algebra.

**Proof.** We indicate the main steps of the proof, which consists of chasing pasting diagrams. To prove the first coherence axiom for the lax algebra $HA$, apply the coherence condition (M1) for $H$, the pseudo-naturality of $h$, the first coherence axiom for the lax algebra $A$, and the modification axiom for $\bar{h}$.

To prove the second coherence axiom for $HA$, apply the coherence condition (M2) for $H$, then the pseudo-naturality of $h$, and finally the second coherence axiom for $A$. The third coherence axiom for $HA$, follows by applying the modification axiom for $\tilde{h}$, the third coherence axiom for the lax algebra $A$, and finally the coherence condition (M3) for $H$.  

### Lemma 3.2.

If $f : A \to B$ is a lax $S$-algebra map, then $Hf : HA \to HB$ is a lax $T$-algebra map. Furthermore, if $f$ is a pseudo-map, so is $Hf$.

**Proof.** To prove the two coherence axioms, apply the modification axioms for $\bar{h}$ and $\tilde{h}$, respectively. The second claim is a consequence of the assumption that $h: TH \to HS$ is a pseudo-natural transformation.

### Lemma 3.3.

If $\alpha : f \to g$ is a $S$-algebra 2-cell, then $H\alpha : Hf \to Hg$ is a $T$-algebra 2-cell.

**Proof.** Apply the pseudo-naturality of $h$.  

### Proposition 3.4.

If $(H, h) : (X, S) \to (Y, T)$ is a lax map of pseudo-monads, then $\hat{H} : \text{Lax}-S\text{-Alg} \to \text{Lax}-T\text{-Alg}$ is a lifting of $H : X \to Y$.

**Proof.** Direct consequence of Lemma 3.1, Lemma 3.2, and Lemma 3.3. The commutativity of the diagram follows by definition of $\hat{H}$.  

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**Diagram:**

![Diagram](image-url)
The corresponding result for pseudo-algebras is deduced as a simple corollary. Recall that if the modifications $\bar{h}$ and $\tilde{h}$ in (2) are invertible, we speak of a pseudo-map of pseudo-monads.

**Corollary 3.5.** If $(H, h) : (X, S) \rightarrow (Y, T)$ is a pseudo-map of pseudo-monads, then the pseudo-functor $\hat{H} : \text{Lax}-S\text{-Alg} \rightarrow \text{Lax}-T\text{-Alg}$ restricts along the inclusions of pseudo-algebras into lax algebras, so as to determine a commutative diagram of the following form.

\[
\begin{array}{ccc}
\text{Ps-S-Alg} & \xrightarrow{\hat{H}} & \text{Ps-T-Alg} \\
\downarrow & & \downarrow \\
\text{Lax-S-Alg} & \xrightarrow{H} & \text{Lax-T-Alg} \\
\downarrow U & & \downarrow U \\
X & \xrightarrow{H} & Y
\end{array}
\]

**Proof.** Immediate consequence of Proposition 3.4. □

Let us now consider a lax pseudo-monad transformation $(p, \bar{p}) : (H, h) \rightarrow (K, k)$. For a lax algebra $A$, the 1-cell $p_A : HA \rightarrow KA$ comes equipped with the 2-cell in the following pasting diagram.

\[
\begin{array}{ccc}
THA & \xrightarrow{Tp_A} & TKA \\
\downarrow h_A & & \downarrow k_A \\
HSA & \xrightarrow{pSA} & KSA \\
\downarrow Ha & & \downarrow Ka \\
HA & \xrightarrow{pA} & KA
\end{array}
\]

The next lemma shows that these pairs can be seen as the components of a natural transformation $\hat{p} : \hat{H} \rightarrow K$.

**Lemma 3.6.** If $A$ is a lax $S$-algebra, then $p_A : HA \rightarrow KA$ is a lax map.

**Proof.** To prove the first coherence axiom for the lax map $p_A$, apply the modification axiom for $\bar{p}$, the pseudo-naturality of $p$, and finally the coherence condition (T1) for the lax pseudo-monad transformation $p$. To establish the second coherence axiom for the lax map $p_A$, apply the coherence condition (T2) for lax pseudo-monad transformation $p$, and then its pseudo-naturality. □

**Proposition 3.7.** If $(p, \bar{p}) : H \rightarrow K$ is a lax pseudo-monad transformation between lax maps of pseudo-monads, then the pseudo-natural transformation $\hat{p} : \hat{H} \rightarrow \hat{K}$ is a lifting of $p : H \rightarrow K$.

**Proof.** Consequence of Lemma 3.6. □

Finally, let $\alpha : (p, \bar{q}) \rightarrow (q, \tilde{q})$ be a pseudo-monad modification between lax pseudo-monad transformations.

**Lemma 3.8.** For a lax $S$-algebra $A$, the 2-cell $\alpha_A : p_A \rightarrow q_A$ is a $T$-algebra 2-cell between lax $T$-algebra maps.

**Proof.** Apply the modification axiom for $\alpha$ and the coherence axiom (MD) for a pseudo-monad modification. □
Proposition 3.9. If $\alpha : p \to q$ is a pseudo-monad modification, then the modification $\hat{\alpha} : \hat{p} \to \hat{q}$ is a lifting of $\alpha : p \to q$.


Proposition 3.4, Proposition 3.7, and Proposition 3.9 imply that we can define a 2-functor $F$ as in (5).

4. The 2-adjunction

We show that the 2-functor $F$ defined in Section 3 determines a 2-adjunction. In order to do so, recall that the lifting $\hat{H} : \text{Lax-S-Alg} \to \text{Lax-T-Alg}$ of a pseudo-functor $H : X \to Y$ defined via a lax pseudo-monad map $(H, h) : (X, S) \to (Y, T)$ has a special property. As stated in Proposition 3.4, the pseudo-functor $\hat{H}$ maps pseudo-maps into pseudo-maps. From now on, we restrict our attention to liftings of this form. In order to avoid introducing further notation, we simply redefine the 2-category $\text{Lift}_{\text{Lax}}[(X, S), (Y, T)]$ to have these special liftings of pseudo-functors as objects. By Proposition 3.4 we can keep writing $F$ for the 2-functor defined in Section 3. Our aim is to obtain a right 2-adjoint $G$ to $F$, so as to determine a 2-adjunction of form

$$P_{\text{sm}} \xleftarrow{F} \text{Lift}_{\text{Lax}}[(X, S), (Y, T)] \xrightarrow{G} \text{Lift}_{\text{Lax}}[(X, S), (Y, T)] .$$

As we will see, the restriction to the special liftings is essential to define the right 2-adjoint.

The right 2-adjoint. Let $\hat{H} : \text{Lax-S-Alg} \to \text{Lax-T-Alg}$ be a lifting of a pseudo-functor $H : X \to Y$. For our purposes we will focus on the lax $T$-algebras associated via $\hat{H}$ to free pseudo-$S$-algebras. An application of $\hat{H}$ to the free pseudo-$S$-algebra on $A$ described in (4) has underlying 0-cell $HA$, structure map $h_A' : THA \to HSA$, associativity and unit 2-cells as below.

For any 1-cell $f : A \to B$ in $X$, we have that $Sf : SA \to SB$ can be equipped with the structure of a pseudo-algebra map, by considering the 2-cell determined by the pseudo-naturality of $m : S^2 \to S$. The assumption that $\hat{H}$ sends pseudo-maps of $S$-algebras to pseudo-maps of $T$-algebras, we get a pseudo-map $Hsf : HSA \to HSB$, which comes equipped with a 2-cell fitting in the diagram below.

The 2-cells $h_f'$ equip the family of 1-cells $h_A' : THA \to HSA$ with the structure of a pseudo-natural transformation $h' : TH \to HS$. We can then define a composite pseudo-natural transformation

$$TH \xrightarrow{TH\alpha} THS \xrightarrow{h'} HS$$
We prove that this pseudo-natural transformation determines a lax map of pseudomonads. In order to do so, it is useful to observe the following fact, concerning the families of 2-cells \( h'_A \) and \( ˜h'_A \), for \( A \in X \), isolated earlier.

**Lemma 4.1.** \( h' \) and \( ˜h' \) are modifications.

**Proof.** Apply the first and second coherence of a lax map, respectively. \( \square \)

Since \( m_A : S^2A \to SA \) can be regarded as a pseudo-map, an application of \( ˜H \) determines a pseudo-map \( Hm_A : HS^2A \to HSA \), and so an invertible 2-cell that fits in the following diagram:

\[
\begin{array}{ccc}
THS^2A & \xrightarrow{THm_A} & THSA \\
\downarrow h'_S & \downarrow H\alpha_A & \downarrow h'_A \\
HS^2A & \xrightarrow{Hm_A} & HSA
\end{array}
\]

The invertibility of this 2-cell allows us to equip the composite pseudo-natural transformation in (7) with appropriate modifications, so as to obtain a lax pseudo-monad map. The modifications for the diagrams in (2) are provided by the following pastings.

\[
\begin{array}{ccccccc}
T^2H & \xrightarrow{T^2Hu} & T^2HS & \xrightarrow{T\alpha'} & THS & \xrightarrow{THuS} & THS^2 \\
\downarrow nH & & \downarrow nHS & & \downarrow \gamma & & \downarrow h'_S \\
TH & \xrightarrow{THu} & THS & & HS^2 & \xrightarrow{Hm} & HS \\
\downarrow H\alpha & & \downarrow 1HS & & \downarrow H\alpha & & \downarrow 1HS \\
& & & & & & H'
\end{array}
\]

In the pasting on the left-hand side, the 2-cell \( \gamma \) is the inverse to the 2-cell obtained from the following pasting of invertible 2-cells:

\[
\begin{array}{ccc}
THS & \xrightarrow{1THS} & THS \\
\downarrow THuS & & \downarrow THu \\
THS^2 & \xrightarrow{THm} & THS \\
\downarrow h'_S & & \downarrow h'_A \\
HS^2 & \xrightarrow{Hm} & HS
\end{array}
\]

(8)

As a consequence of its definition, the 2-cell \( \gamma \) enjoys some coherence properties, which are helpful to prove the next series of propositions.

**Lemma 4.2.** The 2-cell \( \gamma : Hm \cdot h'S \cdot THuS \to h' \) is an algebra 2-cell.

**Proof.** It suffices to show that the inverse of \( \gamma \) is an algebra 2-cell. This follows by the first coherence axiom for the lax algebra maps \( Hm \), and the pseudo-naturality of \( m \). \( \square \)
The arrows labelled simply with an isomorphism in the next lemma can be obtained by appropriate pseudo-naturalities.

**Lemma 4.3.** The following diagrams commute

\[
\begin{align*}
\begin{array}{c}
\text{Id} \\ \downarrow \tilde{h}' \end{array} & \quad \begin{array}{c}
H \rho \\ \downarrow H m \cdot H u S \end{array} \\ & \quad \begin{array}{c}
\tilde{h}' \cdot v H S \\ \downarrow H m \cdot H u S \end{array} \\ & \quad \begin{array}{c}
H m \cdot h' S \cdot v H S^2 \cdot H u S \\ \downarrow \equiv \\
H m \cdot h' S \cdot TH u S \cdot v H S
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
H m \cdot h' S \cdot T H u S \cdot T H u \\ \downarrow \gamma \cdot T H u \\
\end{array} & \quad \begin{array}{c}
\equiv \\
H m \cdot h' S \cdot T H S u \cdot T H u
\end{array}
\]

(10)

**Proof.** To prove the commutativity of the diagram in (9), rephrase the claim as an equivalent statement concerning the inverse of \( \gamma \). At this point, use the second coherence axiom for the lax algebra map \( H m \) and apply the pseudo-naturality of \( u \). For (10), paste the diagram to the 2-cell of which \( \gamma \) is inverse. One composite reduces to the identity by definition of \( \gamma \). The other composite also reduces to the identity using that \( H \lambda \) is an algebra 2-cell and the coherence condition (PSM5) for a pseudo-monad. □

**Proposition 4.4.** \((H, h) : (X, S) \to (Y, T)\) is a lax pseudo-monad map.

**Proof.** To prove the coherence condition (M1), apply the pseudo-naturality of \( u \), the first coherence axiom for the lax algebra \( H S A \), and then apply Lemma 4.2. To prove (M2), use (9), the second coherence axiom for the lax algebra \( H S A \), and finally the second coherence axiom for the lax algebra maps given by the components of the pseudo-natural transformation \( T H u : T H \to T H S \). For (M3), use the modification axiom for \( \lambda \), the third coherence axiom for the lax algebra \( H S A \), and finally (10). □

Let us now consider a lifting \((p, \hat{p}) : (H, \hat{H}) \to (K, \hat{K})\) of a pseudo-natural transformation \( p : H \to K \). We can define a lax transformation \( p : (H, h) \to (K, k)\) of lax pseudo-monad maps by considering the following pasting diagram:

\[
\begin{align*}
\begin{array}{c}
T H \\ \downarrow T H u \end{array} & \quad \begin{array}{c}
T p \\ \downarrow \equiv \\
T K u \end{array} \\ & \quad \begin{array}{c}
T H S \\ \downarrow \gamma \cdot T H u \\
\equiv \\
T K S \end{array}
\end{align*}
\]

The top part of the diagram is an instance of the pseudo-naturality of \( p : H \to K \), and the bottom part is obtained by observing that the 1-cells \( p S_A : H S A \to K S A \) are lax maps of lax algebras.
In the statement of the next lemma, we avoid inserting the required 2-cells. The pasting on the left-hand side is obtained using the pseudo-naturality of \( p, pS^2 \), again the pseudo-naturality of \( p \), and \( \gamma_H \). The pasting on the right-hand side is obtained using \( \gamma_K \) and \( pS \).

**Lemma 4.5.**

\[
\begin{array}{c}
\text{Lemma 4.5.} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Proof.} \text{ Again, replace the statement with an equivalent one involving the inverse to } \gamma_H \text{ and } \gamma_K. \text{ The required claim follows by the coherence axiom for the algebra 2-cell } pS. \quad \square
\end{array}
\]

**Proposition 4.6.** \((p, \bar{p}) : (H, h) \to (K, k)\) is a lax pseudo-monad transformation.

**Proof.** To prove (T1), apply Lemma 4.5, then use the first coherence axiom for the lax algebra map \( pS \), and finally apply the pseudo-naturality of \( p \). The coherence condition (T2) for a lax pseudo-monad transformation follows by applying the pseudo-naturality of \( p \), and then the second coherence axiom for the lax pseudo-monad map \( pS \). \quad \square

Finally, if \((\alpha, \hat{\alpha}) : (p, \hat{p}) \to (q, \hat{q})\) is a lifting of a modification \( \alpha : p \to q \), we can prove the following fact.

**Proposition 4.7.** \( \alpha : p \to q \) is a pseudo-monad modification.

**Proof.** We need to prove (MD). It suffices to apply the coherence axiom for algebra 2-cells and the pseudo-naturality of \( THu : TH \to THS \). \quad \square

Proposition 4.4, Proposition 4.6, and Proposition 4.7 shows that we have a 2-functor \( G \) as in (6). Next, we show that the 2-functors \( F \) and \( G \) are 2-adjoint.

**Unit and counit.** Let us begin by computing the unit and counit of the 2-adjunction. For a lax map of pseudo-monads \((H, h) : (X, S) \to (Y, T)\), we have the lax map \( GF(H, h) \) is given by the pair \((H, h'')\), where \( h'' : TH \to HS \) is the composite pseudo-natural transformation

\[
\begin{array}{c}
\text{Unit and counit.} \\
\end{array}
\]

The component \( \eta_{(H, h)} : (H, h) \to (H, h'') \) of the unit is then the lax pseudo-monad transformation given by the identity pseudo-natural transformation on \( H \), and the
modification obtained via the pasting diagram below, obtained using the pseudo
naturality of \( h : TH \to HS \) and the right unit modification of the pseudo-monad \( S \).

\[
\begin{array}{c}
TH \xrightarrow{Id} TH \\
\downarrow h_u \quad \quad \downarrow h_S \\
\downarrow HS_u \quad \quad \downarrow HS \\
HS \xrightarrow{Id} HS
\end{array}
\]

**Lemma 4.8.** \((Id, \eta_{(H,h)}) : (H,h) \to (H,h'')\) is a lax pseudo-monad transformation. Furthermore, it is invertible.

**Proof.** The invertibility is clear. To prove the first coherence axiom, begin by
inverting the 2-cell playing the role of \( \gamma \) in the pasting diagram. Then use the
pseudo-naturality of \( h \), the modification axiom for the right unit modification \( \rho \) of
the pseudo-monad \( S \), and finally the modification axiom for \( \bar{h} \). To prove the second
coherence axiom for a lax pseudo-monad transformation, apply the modification
axiom for \( \tilde{h} \), and the fourth coherence axiom for a pseudo-monad. \( \Box \)

We now construct the counit of the adjunction. Consider a lifting \((H,\tilde{H})\) of a
pseudo-functor \( H : X \to Y \) to lax algebras, and write \( h' : THS \to HS \) for the
canonical pseudo-natural transformation associated to it, as defined earlier. The
lifting \( GF(H,\tilde{H}) \) is given by the pair \((H,\tilde{H}'')\), where \( \tilde{H}'' \) is the pseudo-functor
mapping a lax \( S \)-algebra \( A \) with structure map \( a : SA \to A \) into the lax \( T \)-algebra
\( HA \) with structure map given by the composite

\[
THA \xrightarrow{THu_a} THSA \xrightarrow{k'_A} HSA \xrightarrow{Ha} HA
\]

In order to provide the counit of the adjunction it is sufficient to give a lifting of
the identity pseudo-natural transformation on \( H \). In order to do so, we define a
family of 2-cells

\[
\begin{array}{c}
THA \xrightarrow{Id_{THA}} THA \\
\downarrow TH\tilde{a} \quad \quad \quad \downarrow H\tilde{a} \\
THSA \xrightarrow{k'_A} HSA \\
\downarrow Ha \quad \quad \quad \downarrow \tilde{a} \\
HA \xrightarrow{Id_{HA}} HA
\end{array}
\]

This pasting is obtained using the unit 2-cell for the algebra \( A \), and the 2-cell that
is obtained by applying \( H \) to the diagram expressing that \( a : SA \to A \) is a lax
map of algebras. We need to show that we have obtained a 1-cell in the 2-category
\( \text{Lift}_{\text{Lax}}[(X,S),(Y,T)] \).
Lemma 4.9. \((\text{Id}_{HA}, \bar{a}) : HA \to HA\) is a lax algebra map.

Proof. For the first coherence axiom, apply the coherence axiom for the algebra 2-cell \(H(\bar{a})\), the third coherence axiom for the lax algebra \(HA\), and then reduce the obtained pasting diagram by observing that we have both the 2-cell \(\gamma\) and its inverse in it. To conclude the proof, apply the first coherence axiom for the lax algebra map \(Ha\), and finally use the pseudo-naturality of \(n\). The second coherence axiom for a lax algebra map follows by application of the second coherence axiom for the lax algebra map \(Ha\), and the pseudo-naturality of \(v\). □

To prove that we have a 2-adjunction it suffices to prove the following lemma.

Lemma 4.10. The unit \(\eta : 1 \to GF\) and counit \(\varepsilon : FG \to 1\) satisfy the triangular laws.

Proof. To verify the first triangular law, stating that the composite

\[
F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F
\]
is the identity, apply the pseudo-naturality of \(h\), and the second coherence axiom for a lax algebra. For the second triangular law, asserting that the composite

\[
G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G
\]
is the identity, use that the image of the left unit modification \(\lambda\) of the pseudo-monad \(S\) under a lifting \(\hat{H}\) is an algebra 2-cell, and then use the second coherence axiom for the pseudo-monad \(S\). □

5. Pseudo-distributive laws

We define pseudo-distributive laws as special lax distributive laws.

Definition 5.1. Let \(S\) and \(T\) be pseudo-monads on a 2-category \(X\). A lax distributive law of \(T\) over \(S\) consists of a pseudo-natural transformation

\[
ST \xrightarrow{d} TS
\]
and four modifications fitting in the following diagrams:
and subject to ten coherence axioms, (D1)-(D10), given in Appendix B. A pseudo-distributive law is a lax distributive law for which the modifications above are invertible.

The coherence axioms for a lax distributive law can be readily explained as follows:

- (D1), (D2), (D3) express that \((T, d) : (X, S) \to (X, S)\) is a lax map of pseudo-monads,
- (D4), (D5) express that \((n, \bar{n}) : (T, d)^2 \to (T, d)\) is a lax pseudo-monad transformation,
- (D6), (D7) express that \((v, \bar{v}) : \text{Id} \to (T, d)\) is a lax pseudo-monad transformation,
- (D8), (D9), and (D10) express respectively that \(\alpha : n \cdot Tn \Rightarrow n \cdot nT\), \(\lambda : n \cdot Tv \Rightarrow \text{Id}\), and \(\rho : \text{Id} \Rightarrow vT \cdot n\) are pseudo-monad modifications.

Here \((T, d)^2 : (X, S) \to (X, S)\) denotes the composition of the lax pseudo-monad map \((T, d) : (X, S) \to (X, S)\) with itself, which can be easily defined. This operation of composition equips the 2-category \(\text{Psm}_{\text{Lax}}[(X, S), (X, S)]\) with a pseudo-monoidal structure [9, 12] that is completely analogous to the one described in [29, Chapter 6] for pseudo-maps, with unit given by the identity map. A pseudo-monoid in this pseudo-monoidal 2-category is precisely a pseudo-monad \(T\) on \(X\) together with a lax distributive of \(T\) over \(S\). We can consider lax distributive laws as the 0-cells of the 2-category \(\text{Dist}_{\text{Lax}}(X, S)\) of pseudo-monoids.

Remark. In the case of a pseudo-distributive laws, the axioms presented above are equivalent to the ones given in [29]. The detailed comparison of equivalences is as follows: (D1) \(\sim (T6)\), (D2) \(\sim (T2)\), (D3) \(\sim (T3)\), (D4) \(\sim (T8)\), (D5) \(\sim (T1)\), (D6) \(\sim (T10)\), (D7) \(\sim (T9)\), (D8) \(\sim (T7)\), (D9) \(\sim (T4)\), and finally (D10) \(\sim (T5)\). The comparison with the nine coherence axioms for pseudo-distributive laws in [25] is as follows: (D1) \(\sim (\text{coh 4})\), (D2) \(\sim (\text{coh 2})\), (D4) \(\sim (\text{coh 3})\), (D5) \(\sim (\text{coh 1})\), (D6) \(\sim (\text{coh 6})\), (D7) \(\sim (\text{coh 5})\), (D8) \(\sim (\text{coh 9})\), (D9) \(\sim (\text{coh 8})\), and finally (D10) \(\sim (\text{coh 7})\).

In order to relate lax distributive laws and liftings to lax algebras of pseudo-monads, it is convenient to observe that also the 2-category \(\text{Lift}_{\text{Lax}}[(X, S), (X, S)]\) has the structure of a pseudo-monoidal 2-category, again given by composition. This time, a pseudo-monoid is a pseudo-monad \((X, T)\) with a lifting to \(\text{Lax-S-Alg}\), by which we obviously mean a lifting of each of the data determining the pseudo-monad \((X, T)\). We then write \(\text{Lift}_{\text{Lax}}(X, S)\) for the 2-category of pseudo-monoids.

**Theorem 5.2.** The 2-adjunction

\[
\begin{align*}
\text{Psm}_{\text{Lax}}[(X, S), (X, S)] & \xrightarrow{\perp} \text{Lift}_{\text{Lax}}[(X, S), (X, S)] \\
\text{Dist}_{\text{Lax}}(X, S) & \xrightarrow{\perp} \text{Lift}_{\text{Lax}}(X, S) \\
\text{Psm}_{\text{Lax}}[(X, S), (X, S)] & \xrightarrow{\perp} \text{Lift}_{\text{Lax}}[(X, S), (X, S)]
\end{align*}
\]

lifts to a 2-adjunction between lax distributive laws and liftings of pseudo-monads to lax algebras, in the sense that there is a diagram of form

where the vertical arrows are the obvious forgetful 2-functors.
Proof. The result follows once we show that the 2-adjunction in (11) is lax pseudo-monoidal. By a generalisation of the results in [9], which in turn generalise those in [17], it is sufficient to prove that the left adjoint is a strong pseudo-monoidal 2-functor. This can be shown by a straightforward generalisation of the proofs in [29]. □

The next corollary, which expresses the equivalence between pseudo-distributive laws and liftings to 2-categories of pseudo-algebras, strengthens the main result of [29], and puts it in a more general context.

**Corollary 5.3.** The adjoint 2-equivalence

\[ \text{Psm}[(X, S), (X, S)] \xrightarrow{\hat{F}} \text{Lift}[(X, S), (X, S)] \]

lifts to an adjoint 2-equivalence between pseudo-distributive laws and liftings of pseudo-monads to Ps-S-Alg, in the sense that there is a diagram of form

\[ \text{Dist}(X, S) \xrightarrow{\hat{F}} \text{Lift}(X, S) \]

\[ \text{Psm}[(X, S), (X, S)] \xrightarrow{\hat{G}} \text{Lift}[(X, S), (X, S)] \]

We can now describe why the tenth coherence axiom for pseudo-distributive law introduced in [29] is not necessary to obtain Corollary 5.3. First of all, to define a lifting to pseudo-algebras \( \hat{H} : \text{Ps-S-Alg} \to \text{Ps-T-Alg} \) of a pseudo-functor \( H : X \to Y \) from a pseudo-map of pseudo-monad, the coherence condition (M3) for pseudo-monad maps is not necessary. Its only application, in the proof of Lemma 3.1, is in the proof of a coherence condition that is redundant for pseudo-algebras. This reasoning can be applied also to liftings of pseudo-monads, rather than pseudo-functors. The coherence condition (D3) expresses (M3) for \((T, d) : (X, S) \to (X, S)\), and therefore it is not necessary to show that the pseudo-functor part of the pseudo-monad \( T \) can be lifted to Ps-S-Alg.

### 6. Conclusions

A number of examples of liftings of the presheaf construction to 2-categories of algebras will be given in [11] as part of the general theory of two-dimensional Kleisli structures. The notion of a Kleisli structure on a 2-category, which is essentially a two-dimensional version of Manes’ presentation of a monad [22], generalises the notion of a pseudo-monad so as to allow us to capture the presheaf construction as a natural example without invoking delicate set-theoretical considerations.

It is then a lengthy but essentially straightforward exercise to rephrase the coherence conditions for a lax distributive law between pseudo-monads and define the notion of a lax distributive law between a Kleisli structure and a pseudo-monad. Typical examples of liftings of the presheaf construction are then readily provided by known results on the convolution tensor product [8, 14]. Applications of this theory have already been considered in [6]. Furthermore, in [10] it will shown how the pseudo-distributive law existing between the presheaf construction and the 2-monad for symmetric strict monoidal categories can be applied to construct the cartesian closed bicategory of generalised species of structures.
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Appendix A. Coherence for pseudo-monads

Pseudo-monads. These axioms refer to the data for a pseudo-monad \((X, S)\), as given in (1).

\[
\begin{align*}
\text{(PSM1)} & \quad S^4 \xrightarrow{S^2 m} S^3 \\
& \quad S^3 \xrightarrow{S m} S^2 \\
& \quad S^2 \xrightarrow{S m} S \\
& \quad m \xrightarrow{m} 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{(PSM2)} & \quad S^2 \xrightarrow{1_{S^2}} S^2 \\
& \quad S^3 \xrightarrow{S m} S^2 \\
& \quad S^2 \xrightarrow{1_{S^2}} S \\
& \quad m \xrightarrow{m} 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{(PSM3)} & \quad S^2 \xrightarrow{m} S \\
& \quad S^3 \xrightarrow{S m} S^2 \\
& \quad S^2 \xrightarrow{1_{S^2}} S \\
& \quad S^2 \xrightarrow{1_{S^2}} S \\
& \quad m \xrightarrow{m} 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{(PSM4)} & \quad 1 \xrightarrow{u} S \\
& \quad u \xrightarrow{u_{S^2}} S u \\
& \quad u \xrightarrow{u_{S^2}} S^2 \\
& \quad S^2 \xrightarrow{1_S} S \\
& \quad S \xrightarrow{1_S} S \\
\end{align*}
\]
Remark. Given the complexity of the following pasting diagrams, we limit ourselves to draw the boundaries of the diagrams involved, and invite the reader to fill them with the relevant 2-cells, which we describe with comments.

**Lax pseudo-monad maps.** These axioms refer to a lax pseudo-monad map $(H, h) : (X, S) \to (Y, T)$, where $H : TH \to HS$ and with modifications as in (2).
The left-hand side pasting is obtained with $T\tilde{h}$, $\tilde{h}$, and the associativity of the pseudo-monad $T$. The right-hand side pasting is obtained instead with the pseudonaturality of $h$, the associativity of the pseudo-monad $S$, $hS$, pseudo-naturality of $n$, and $\tilde{h}$.

The left-hand side pasting is obtained with $T\tilde{h}$, $\tilde{h}$, and the associativity of the pseudo-monad $T$. The right-hand side pasting is obtained instead with the pseudonaturality of $h$, the associativity of the pseudo-monad $S$, $hS$, pseudo-naturality of $n$, and $\tilde{h}$.

The left-hand side pasting is obtained with $T\tilde{h}$, $\tilde{h}$, and the left unit of the pseudo-monad $T$. The right-hand side pasting is obtained with the pseudonaturality of $h$ and the left unit of the pseudo-monad $S$.

The left-hand side pasting is obtained with the right unit law of the pseudo-monad $S$, $\tilde{h}S$, the pseudonaturality of $v$, and $\tilde{h}$. The right-hand side pasting is obtained with the right unit of the pseudo-monad $T$.

**Lax pseudo-monad transformation.** The coherence axioms refer to a pair $(p, \bar{p}) : (H, h) \rightarrow (K, k)$ as in (3), where $(H, h)$ and $(K, k)$ are lax pseudo-monad maps.
The left-hand side pasting is obtained with $T \bar{p}$, $\bar{p}S$, the pseudo-naturality of $p$, and $\bar{h}$. The right-hand side pasting is obtained with $\bar{k}$, the pseudo-naturality of $n$, and $\bar{p}$.

\[ (T1) \]

The left-hand side pasting is obtained with $\tilde{h}$ and the pseudo-naturality of $p$. The right-hand side pasting is obtained with $\tilde{k}$, the pseudo-naturality of $v$, and $\bar{p}$.

**Pseudo-monad modification.** This coherence condition refers to a modification $\alpha : p \to q$, where $(p, \bar{p})$ and $(q, \bar{q})$ are pseudo-monad transformations.

\[ (MD) \]

The left-hand side pasting is obtained with $T \alpha$ and $\bar{q}$. The right-hand side pasting is obtained with $\bar{p}$ and $\alpha S$. 
The left-hand side pasting is obtained using $S\bar{m}$, $\bar{m}$, and the associativity of the pseudo-monad $S$. The right-hand side pasting is obtained using the pseudo-naturality of $d$, the associativity of the pseudo-monad $S$, the pseudo-naturality of $m$, and $\bar{m}$.
The left-hand side pasting is obtained using $S\bar{u}$, $\bar{m}$, and the left unit of the pseudo-monad $S$. The right-hand side pasting is obtained using the pseudo-naturality of $d$ and the left unit of the pseudo-monad $S$.

The left-hand side pasting is obtained using the right unit of the pseudo-monad $S$, $\bar{u}$, the pseudo-naturality of $u$, and $\bar{m}$. The right-hand side pasting is obtained using the right unit of the pseudo-monad $S$.

The left-hand side pasting is obtained using $S^2v$, and the left unit of the pseudo-monad $S$. The right-hand side pasting is obtained using $S^2v$.
The left-hand side pasting is obtained using $S\bar{v}$, $\bar{v}S$, and the pseudo-naturality of $v$. The right-hand side is obtained using $\bar{m}$, the pseudo-naturality of $m$, and $\bar{m}$.

(D5)

The left-hand side pasting is obtained using $\bar{u}$, the pseudo-naturality of $u$, and $\bar{v}$. The right-hand side pasting is the pseudo-naturality of $\bar{v}$. 
The left-hand side pasting is obtained using $S\bar{n}$, $\bar{n}S$, the pseudo-naturality of $n$, the pseudo-naturality of $d$, $T\bar{m}$, and $\bar{m}T$. The right-hand side pasting is obtained using $\bar{m}$, the pseudo-naturality of $m$, and $\text{mut}$.
The left-hand side pasting is obtained using \( \bar{u} \), the pseudo-naturality of \( u \), and \( \bar{n} \). The right-hand side pasting is obtained using the pseudo-naturality of \( n \), \( T\bar{u} \), and \( \bar{u}T \).

To state the next three coherence conditions for a lax distributive laws, let \( \alpha \), \( \lambda \), and \( \rho \) be the associativity, left unit, and right unit for the pseudo-monad \( T \).

The left-hand side pasting is obtained using \( S\alpha \), \( \bar{n} \), \( \bar{n}T \), and the pseudo-naturality of \( n \). The right-hand side pasting is obtained using \( \bar{n} \), the pseudo-naturality of \( d \), \( T\bar{n} \), and \( \alpha S \).
The left-hand side pasting uses $S\lambda$. The right-hand side pasting is obtained using $\bar{n}$, the pseudo-naturality of $d$, $\bar{v}$, and $\lambda S$.

The left-hand side pasting is obtained using $S\rho$, $\bar{n}$, $\bar{v}S$, and the pseudo-naturality of $v$. The right-hand side pasting is $\rho S$.

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