HOMOTOPY LIMITS FOR 2-CATEGORIES

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Abstract. We study homotopy limits for 2-categories using the theory of Quillen model categories. In order to do so, we establish the existence of projective and injective model structures on diagram 2-categories. Using this result, we describe the homotopical behaviour not only of conical limits but also of weighted limits for 2-categories. Finally, homotopy limits are related to pseudo-limits.

1. Quillen model structures in 2-category theory

The 2-category of groupoids, functors, and natural transformations admits a model structure in which the weak equivalences are the equivalence of categories and the fibrations are the Grothendieck fibrations [1, 5, 13]. Similarly, the 2-category of small categories, functors, and natural transformations admits a model structure in which the weak equivalences are the equivalence of categories and the fibrations are the isofibrations, which are functors satisfying a restricted version of the lifting condition for Grothendieck fibrations which involves only isomorphisms [13, 19]. Steve Lack has vastly generalised these results by showing that every 2-category $\mathcal{K}$ with finite limits and colimits admits a model structure, called here the *natural model structure* on $\mathcal{K}$, in which the weak equivalences and the fibrations are the equivalences and the isofibrations in $\mathcal{K}$ [17]. The notions of equivalence and isofibration for a map in a 2-category are obtained by suitably generalising the notions of equivalence and of isofibration for a functor. We take Lack's theorem as a starting point to study homotopy limits for 2-categories.

Our first step is to show that for every small 2-category $\mathcal{A}$ and every 2-category $\mathcal{K}$ with finite limits and small colimits, the functor 2-category $[\mathcal{A}, \mathcal{K}]$ admits a model structure in which the weak equivalences are the pointwise equivalences and the fibrations are the pointwise isofibrations. We refer to this model structure as the *projective model structure*. When $\mathcal{K}$ is assumed to be locally presentable, the existence of the projective model structure follows by a result on the lifting of the natural model structure on a 2-category $\mathcal{K}$ to 2-categories of algebras for a 2-monad with rank on $\mathcal{K}$ [17, Theorem 4.5]. The special form of the 2-category $[\mathcal{A}, \mathcal{K}]$, however, allows us to avoid assuming that $\mathcal{K}$ is locally presentable, and to give a simple proof of the model category axioms for the projective model structure, which does not

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make any direct use of transfinite induction and provides explicit methods to produce the required factorisations and liftings.

We observe that by duality every 2-category $\mathcal{K}$ with finite limits and colimits admits a dual of its natural model structure, in which the weak equivalences are the categorical equivalences and the cofibrations are the isocofibrations, that is to say the maps which are isofibrations in the 2-category $\mathcal{K}^{\mathrm{op}}$, obtained by reversing maps, but not 2-cells, of $\mathcal{K}$. We will then show that every small 2-category $\mathcal{A}$ and every 2-category $\mathcal{K}$ with small limits and finite colimits, the functor 2-category $[\mathcal{A}, \mathcal{K}]$ admits a model structure in which the weak equivalences are the pointwise equivalences and the cofibrations are the pointwise isocofibrations. We refer to this model structure as the *injective model structure*.

The existence of the projective and injective model structures allows us to apply the general theory of enriched model categories [7, 12, 20] to study the total derived functors of limit 2-functors. We will consider not only conical limits but also weighted limits [15, 16, 22]. The study of homotopy-theoretic aspects of weighted limits for 2-categories reveals that there are two different combinations of model structures on 2-categories of diagrams that allow us to regard the weighted limit 2-functor as a right Quillen 2-functor in two variables. This observation gives rise to two different, but equivalent, ways of describing weighted homotopy limits in terms of weighed limits.

In order to describe precisely the completeness properties of many 2-categories of interest, such as those of categories equipped with algebraic structure [4], there is a rich theory of 2-categorical limits [3, 16, 18, 22]. We will relate homotopy limits to pseudo-limits [16, 22]. We do so not only at the level of universal properties, but also by showing how the two reductions of weighted homotopy limits to weighted limits correspond to two ways of reducing the weighted pseudo-limits to weighted limits. An investigation of homotopy limits in the general context of enriched homotopy theory is developed in [21].

2. Model 2-categories

2.1. **The natural model structure on Cat.** The category **Cat** of small categories and functors admits a model structure in which the weak equivalences are the fully faithful and essentially surjective functors, and the cofibrations are the functors injective on objects [13]. A functor $f : A \to B$ is a *categorical equivalence* if there exist a functor $g : B \to A$ and natural isomorphisms $\eta : 1_A \Rightarrow gf$ and $\varepsilon : gf \Rightarrow 1_B$. A functor $f : A \to B$ is said to be an *isofibration* if for every $a \in A$, $b \in B$ and isomorphism $\beta : fa \to b$, there exists a lifting of β, given by an $a' \in A$ and an isomorphism $\alpha : a \to a'$ such that $fa' = b$ and $f\alpha = \beta$. A *cleavage* for an isofibration $f : A \to B$ is a function assigning a lifting to each isomorphism $\beta : f(a) \to b$ in B. As a special case of [17, Theorem 3.3], the category **Cat** admits also a model structure in which the weak equivalences are the categorical equivalences and the fibrations are the isofibrations for which there exists a cleavage. Assuming the Axiom of Choice, which is not necessary to establish these results, the two model structures coincide. In particular, every fully faithful and essentially surjective functor is a categorical equivalence and every isofibration admits a cleavage. We refer to this model structure as the *natural model structure* on **Cat**. We write Ho(**Cat**) for the homotopy category of **Cat**, which consists of categories and isomorphism classes of functors, and denote the localization functor as $\lambda : \mathbf{Cat} \to \mathrm{Ho}(\mathbf{Cat})$. As recalled in [17, §2.2], the cartesian product equips **Cat** with a symmetric monoidal structure which satisfies the axioms for a monoidal model category [7, 12, 20].

2.2. **Model 2-categories.** For any monoidal model category $\mathcal{V}$, there is an associated notion of a model $\mathcal{V}$-enriched category [7, 12]. We spell out the general definition in the special case when $\mathcal{V}$ is **Cat**. In order to do so, for a 2-category $\mathcal{K}$, let us write $\mathcal{K}(X, Y)$ for the hom-category associated to a pair of objects $X, Y \in \mathcal{K}$. A pair of maps $u : X \to Y$ and $v : V \to W$ in $\mathcal{K}$ determines the commutative diagram of categories and functors

$$\begin{array}{ccc} \mathcal{K}(Y,V) & \xrightarrow{\mathcal{K}(u,V)} & \mathcal{K}(X,V) \\ {\scriptstyle \mathcal{K}(Y,v)}\downarrow & & \downarrow{\scriptstyle \mathcal{K}(X,v)} \\ \mathcal{K}(Y,W) & \xrightarrow[\mathcal{K}(u,W)]{} & \mathcal{K}(X,W) \end{array} \tag{1}$$

Since **Cat** has pullbacks, we obtain a canonical functor, denoted

$$[u, v] : \mathcal{K}(Y, V) \to \mathcal{K}(Y, W) \times_{\mathcal{K}(X,W)} \mathcal{K}(X, V) \tag{2}$$

This map is used in Definition 2.2.1.

2.2.1. **Definition.** Let $\mathcal{K}$ be a 2-category with finite limits and colimits. A *model* 2*-structure* on $\mathcal{K}$ consists of a model structure on its underlying category such that the following condition holds: if $u : X \to Y$ is a cofibration and $v : V \to W$ is a fibration in $\mathcal{K}$, then the functor $[u, v]$ is an isofibration in **Cat**, which is a categorical equivalence whenever either u or v is a weak equivalence. A *model* 2*-category* is a 2-category with finite limits and colimits which is equipped with a model 2-structure.

For a 2-category $\mathcal{K}$, we write $\mathcal{K}^{\mathrm{op}}$ for the category obtained from $\mathcal{K}$ by formally reversing the direction of the maps of $\mathcal{K}$, but leaving the 2-cells unchanged. The function mapping a pairs of objects $X, Y \in \mathcal{K}$ to the category $\mathcal{K}(X, Y)$ determines a 2-functor $\mathcal{K}(-, -) : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathbf{Cat}$. We say that $\mathcal{K}$ has *tensors* if for every $X \in \mathcal{K}$ the 2-functor $\mathcal{K}(X, -) : \mathcal{K} \to \mathbf{Cat}$ has a left 2-adjoint. The left 2-adjoint sends $A \in \mathbf{Cat}$ into $A \otimes X \in \mathcal{K}$, the A-tensor of X, and the 2-adjointness means that we have a 2-natural isomorphism with components

$$\mathcal{K}\big(A \otimes X, Y\big) \cong \mathbf{Cat}\big(A, \mathcal{K}(X, Y)\big)\,. \tag{3}$$

Similarly, we say that $\mathcal{K}$ has *cotensors* if for every $Y \in \mathcal{K}$, the functor $\mathcal{K}(-, Y) : \mathcal{K}^{\mathrm{op}} \to \mathbf{Cat}$ has a left adjoint. The left 2-adjoint sends $A \in \mathbf{Cat}$ into $Y^A \in \mathcal{K}$, the A-cotensor of Y. Here, the 2-adjointness means that we have a 2-natural isomorphism with components

$$\mathcal{K}\big(X, Y^A\big) \cong \mathbf{Cat}\big(A, \mathcal{K}(X, Y)\big) . \tag{4}$$

It is convenient to have equivalent characterizations of the notion of a model 2-category under the assumption that $\mathcal{K}$ is tensored or cotensored. This involves the construction of analogues of the map defined in (2). Let us consider maps $f : A \to B$ in $\mathbf{Cat}$ and $u : X \to Y$ in $\mathcal{K}$. When $\mathcal{K}$ has tensors, there is an evident analogue of the diagram in (1), and the universal property of pushouts gives us canonical map

$$\langle f, u \rangle : (A \otimes X) \sqcup_{A \otimes X} (B \otimes X) \to B \otimes Y .$$

When $\mathcal{K}$ has cotensors, the universal property of pullbacks gives us a canonical map

$$\{f, u\} : X^B \to Y^B \times_{Y^A} X^A .$$

Lemma 2.2.2 is a special case of [7, Proposition 3.4]. It will be useful in the study of homotopy-theoretic aspects of 2-categorical limits.

2.2.2. **Lemma.** *Let $\mathcal{K}$ be a 2-category with finite limits whose underlying category is equipped with a model structure.*

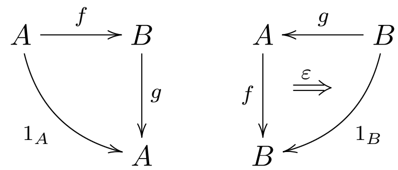
(i) If $\mathcal{K}$ is tensored, $\mathcal{K}$ is a model 2-category if and only if the following conditions hold: if f is a cofibration in $\mathbf{Cat}$ *and u is a fibration in $\mathcal{K}$, then $\langle f, u \rangle$ is a cofibration in $\mathcal{K}$, which is a weak equivalence whenever either f or u is so.*

(ii) If $\mathcal{K}$ is cotensored, $\mathcal{K}$ is a model 2-category if and only if the following condition holds: if f is a cofibration in $\mathbf{Cat}$ *and u is a fibration in $\mathcal{K}$, then $\{f, u\}$ is a fibration in $\mathcal{K}$ which is a weak equivalence whenever either f or u is so.*

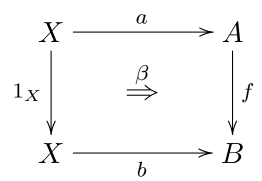
While the definition of a model 2-category involves a compatibilty condition between different model structures, the notion of a Quillen 2-adjunction can be formulated in a straightforward fashion, simply recalling that a 2-adjunction between 2-categories determines an adjunction between their underlying categories. A *Quillen 2-adjunction* between model 2-categories consists of a 2-adjunction whose underlying adjunction is a Quillen adjunction [12, §1.3.1]. The notion of *Quillen 2-equivalence* is defined analgously, using the familiar notion of a Quillen equivalence [12, §1.3.3]. The notion of a Quillen 2-functor in two variables is discussed in Section 5.1.

2.3. **The natural model structure on a 2-category.** The notions of categorical equivalence and of isofibrations recalled in Section 2.1 can be expressed not only in $\mathbf{Cat}$, but within any 2-category $\mathcal{K}$ as follows. We call a map $f : B \to A$ in $\mathcal{K}$ an *equivalence* if there exists a map $g : A \to B$ and invertible 2-cells $\eta : 1_A \Rightarrow gf$ and $\varepsilon : fg \Rightarrow 1_B$. We refer to such $g : B \to A$

as a *quasi-inverse* of $f : A \to B$. If an equivalence has a section, then it is called a *surjective equivalence*; if it has a retraction then it is called an *injective equivalence*. For example, given diagrams of form

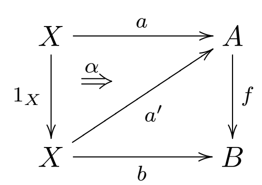


where ε is an invertible 2-cell, then f is an injective equivalence and g is a surjective equivalence. Following [17, §3.4], we define a map $f : A \to B$ to be an *isofibration* if for every diagram of form



(5)

where $\beta : b \Rightarrow fa$ is an invertible 2-cell, there exists a map $a' : X \to A$ such that $b = fa'$ and an invertible 2-cell $\alpha : a' \Rightarrow a$ such that $\beta : b \Rightarrow fa$ equals the composite 2-cell $f\alpha : b \Rightarrow fa$, obtained by the following pasting diagram



(6)

As shown in [17, Section 3] every 2-category $\mathcal{K}$ with finite limits and colimits admits a model 2-structure in which the weak equivalences are the equivalences and the fibrations are the isofibrations. We refer to this model structure as the *natural model structure* on $\mathcal{K}$. The acyclic fibrations in the natural model structure are the surjective equivalences. The special case of this result for the 2-category **Cat** of small categories, functors, and natural transformations gives back the natural model structure discussed in Section 2.1 and implies that it is a model 2-category structure.

3. Projective model structures on 2-categories of diagrams

3.1. **Pointwise equivalences.** For a small 2-category $\mathcal{A}$, we often refer to 2-functors $F : \mathcal{A} \to \mathcal{K}$ as *diagrams*. We write $[\mathcal{A}, \mathcal{K}]$ for the 2-category of diagrams, 2-natural transformations, and modifications. This 2-category has again finite limits and colimits, since all limits and colimits are computed pointwise [15, 16]. We say that a 2-natural transformation $m : F \to G$, where F and G are diagrams, is a *pointwise equivalence* if all its components $m_A : FA \to GA$ are equivalences in $\mathcal{K}$. In general, a pointwise equivalence is not an equivalence in $[\mathcal{A}, \mathcal{K}]$. However, as we recall below, the results in [14] imply that a pointwise equivalence is an equivalence in the larger 2-category of diagrams, pseudo-natural transformations, and modifications, that we denote $\mathrm{Psd}[\mathcal{A},\mathcal{K}]$. Let us recall that the notion of a pseudo-natural transformation generalises that of a 2-natural transformations by allowing the 2-naturality squares to commute up to coherent isomorphism rather than strictly. More precisely, given diagrams F and G, a pseudo-natural transformation $m : F \to G$ consists of a family of maps $m_A : FA \to GA$ in $\mathcal{K}$, for $A \in \mathcal{A}$, and a family of invertible 2-cells m_u, for $u : A \to B$ in $\mathcal{A}$, fitting in diagrams of form

$$\begin{array}{ccc} FA & \xrightarrow{m_A} & GA \\ {\scriptstyle F(u)}\downarrow & \Downarrow m_u & \downarrow{\scriptstyle G(u)} \\ FB & \xrightarrow[m_B]{} & GB \end{array} \tag{7}$$

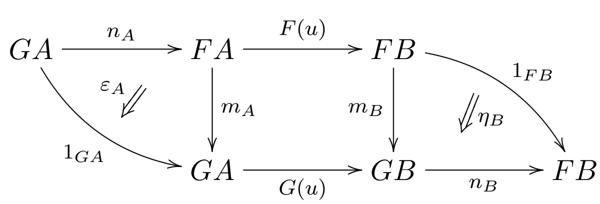
These 2-cells are subject to coherence axioms [22, §4] that express suitable compatibility conditions with respect to identity and composition of maps in $\mathcal{A}$, and with composition with 2-cells in $\mathcal{A}$. A 2-natural transformation can be seen as a pseudo-natural transformation and therefore there is an inclusion 2-functor

$$[\mathcal{A},\mathcal{K}] \xrightarrow{\quad J \quad} \mathrm{Psd}[\mathcal{A},\mathcal{K}] \,. \tag{8}$$

If $m : F \to G$ is a a pointwise equivalence, then there exists a pseudo-natural transformation $n : G \to F$ that is a pointwise quasi-inverse for $m : F \to G$. Indeed, for $A \in \mathcal{A}$, let $n_A : G(A) \to F(A)$ be a quasi-inverse to $m_A : F(A) \to G(A)$ in $\mathcal{K}$. We can assume without loss of generality that m_A and n_A form an adjoint equivalence in $\mathcal{K}$,

$$FA \underset{n_A}{\overset{m_A}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} GA \,. \tag{9}$$

Let us write $\eta_A : 1_{FA} \Rightarrow n_A m_A$ and $\varepsilon_A : m_A n_A \Rightarrow 1_{GA}$ for the invertible 2-cells providing the unit and counit of the adjoint equivalence. respectively. Using them, we can equip the family $n_A : GA \to FA$ with 2-cells so as to obtain a pseudo-natural transformation $n : G \to F$. For $u : A \to B$ in $\mathcal{A}$, we define a 2-cell $n_u : F(u)\, n_A \Rightarrow n_B\, G(u)$ as the composite of the invertible 2-cells appearing in the following diagram:



With this definition, the coherence axioms for a pseudo-natural transformations follow easily from the triangular laws of the adjoint equivalence in (9).

3.2. **The projective model structure.** Let $\mathcal{A}$ be a small category, and let $\mathcal{K}$ be a model 2-category. A 2-natural transformation $m : F \to G$ between diagrams is said to be a *pointwise weak equivalence* if all of its components $m_A : FA \to GA$ are weak equivalences in the model 2-structure on $\mathcal{K}$. The notions of *pointwise fibration* and *pointwise cofibration* are defined analogously. We say that a 2-natural transformation is a *projective cofibration* if it has the left lifting property with respect to the 2-natural transformations which are pointwise acyclic fibrations. Relative to the model 2-category structure on $\mathcal{K}$, the projective model structure on the 2-category $[\mathcal{A}, \mathcal{K}]$ is defined as follows:

$$[\mathcal{A}, \mathcal{K}]_{\mathrm{Proj}} = \begin{cases} \text{Weak equivalences} & = \text{pointwise weak equivalences,} \\ \text{Fibrations} & = \text{pointwise fibrations,} \\ \text{Cofibrations} & = \text{projective cofibrations.} \end{cases}$$

In general, it is not known whether these definitions satisfy Quillen's axioms for a model category. Let us now consider a 2-category $\mathcal{K}$ with finite limits and colimits, and regard it as equipped with its natural model 2-structure. We will show that if $\mathcal{K}$ is cocomplete, then then the diagram 2-category $[\mathcal{A}, \mathcal{K}]$ admits the projective model structure. Let us emphasize that the projective model structure we will establish is relative to the natural model structure on $\mathcal{K}$. Therefore, a pointwise weak equivalence is a pointwise equivalence, a pointwise fibration is a pointwise isofibration, and so a pointwise acyclic fibration is a pointwise surjective equivalence. Furthermore, every diagram will be projectively fibrant. The proof of Lemma 3.2.1 uses the observations on pseudo-natural transformations made in Section 3.1.

3.2.1. **Lemma.** *Let $m : F \to G$ be a 2-natural transformation that is a pointwise cofibration, and let $n : H \to K$ be a 2-natural transformation that is a pointwise fibration. If either m or n is a pointwise weak equivalence, then for every commutative square in* $\mathrm{Psd}[\mathcal{A}, \mathcal{K}]$

$$\begin{array}{ccc} F & \xrightarrow{s} & H \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ G & \xrightarrow[t]{} & K \end{array}$$

there exists a pseudo-natural transformation $j : G \to H$ that is a filler for the diagram.

Proof. Let $m : F \to G$ be a pointwise equivalence, and so a pointwise acyclic cofibration. Since $n : H \to K$ is a pointwise fibration, for every $A \in \mathcal{A}$ there exists a filler for the commutative diagram

$$\begin{CD} FA @>{s_A}>> HA \\ @V{m_A}VV @VV{n_A}V \\ GA @>>{t_A}> KA \end{CD}$$

Let $j_A : GA \to HA$ be a filler. Since $m : F \to G$ is a pointwise equivalence, we can use its pseudo-natural quasi-inverse to construct 2-cells making the maps $j_A : GA \to HA$ into a pseudo-natural transformation. The case when $n : H \to K$ is a pointwise acyclic fibration is treated analogously. $\square$

3.2.2. **Lemma.** *Every* 2*-natural transformation can be factored both as a pointwise cofibration followed by a pointwise acyclic fibration, and as a pointwise acyclic cofibration followed by a pointwise fibration.*

Proof. The factorizations in $\mathcal{K}$ are functorial. $\square$

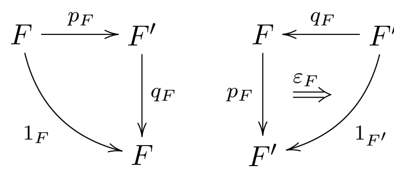
We now assume that $\mathcal{K}$ is also cocomplete, so as to be able to apply the results in [4]. Recall that we write $\mathrm{Psd}[\mathcal{A}, \mathcal{K}]$ for the 2-category of diagrams, pseudo-natural transformations, and modifications. A crucial fact underlying our verification of Quillen's axioms for the projective model structure on $[\mathcal{A}, \mathcal{K}]$ is the consequence of [4, Theorem 3.16] exhibiting a 2-adjunction of form

$$[\mathcal{A}, \mathcal{K}] \underset{J}{\overset{(-)'}{\underset{\bot}{\leftrightarrows}}} \mathrm{Psd}[\mathcal{A}, \mathcal{K}] \, . \tag{10}$$

The right 2-adjoint J is the inclusion 2-functor in (8). Since J is an inclusion, it will not be mentioned explicitly in the following. The left 2-adjoint maps a diagram F into a diagram F', called the *flexible diagram* associated to F. The components of the unit of the 2-adjunction are pseudo-natural transformations $p_F : F \to F'$ which are universal in the sense that every pseudo-natural transformation $m : F \to G$ factors uniquely through $p_F : F \to F'$ in a diagram of form

$$\begin{array}{ccc} F & \xrightarrow{p_F} & F' \\ & \underset{m}{\searrow} & \Big\downarrow{\scriptstyle \bar{m}} \\ & & G \end{array}$$

The components of the counit are 2-natural transformations $q_F : F' \to F$. As shown within the theory of 2-monads in [4, Theorem 4.2] and explained in the special case of interest to us in [3, §4], the pseudo-natural transformations p_F and the 2-natural transformations q_F form an adjoint equivalence. In particular, we have diagrams



(11)

The 2-cell $\varepsilon_F : q_F\, p_F \Rightarrow 1_{F'}$ is the invertible modification providing the counit of the adjoint equivalence. Lemma 3.2.3 states an important property of the counit of the 2-adjunction in (10). As we will see in Remark 3.2.7, the flexible 2-functor associated to a 2-functor can be seen as its cofibrant replacement with respect to the projective model structure on $[\mathcal{A}, \mathcal{K}]$.

3.2.3. **Lemma.** *For every F, the 2-natural transformation $q_F : F' \to F$ is a pointwise acyclic fibration.*

Proof. We need to show that $q_F : F' \to F$ is a pointwise surjective equivalence in $\mathcal{K}$. This follows by instanciating pointwise the diagrams in (11). ☐

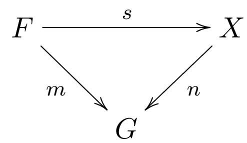
For the statement of Lemma 3.2.4, recall that a 2-natural transformation is a *projective cofibration* if it has the left lifting property with respect to the pointwise acyclic fibrations. We also say that a 2-natural transformation is a *projective trivial cofibration* if it has the left lifting property with respect to the pointwise fibrations.

3.2.4. **Lemma.** *If $m : F \to G$ is a pointwise cofibration, then $m' : F' \to G'$ is a projective cofibration. Furthermore, when m is a pointwise acyclic cofibration, then m' is a projective trivial cofibration.*

Proof. Assume that m is a pointwise cofibration. It is necessary to show that m' has the left lifting property with respect to pointwise acyclic fibrations. In order to do so, first use the 2-adjunction in (10) to transfer the lifting problem from $[\mathcal{A}, \mathcal{K}]$ to $\mathrm{Psd}[\mathcal{A}, \mathcal{K}]$, and then apply Lemma 3.2.1. The proof of the other claim is analogous. ☐

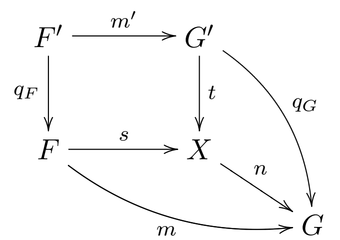
Lemma 3.2.5 is the crucial step to establish that the projective model structure satisfies the axioms for a Quillen model structure. In its proof, we use the 2-adjunction in (10).

3.2.5. **Lemma.** *Every pointwise cofibration $m : F \to G$ can be factored as follows*



where $s : F \to X$ is a projective cofibration, and $n : X \to G$ is a pointwise acyclic fibration. Furthermore, if $m : F \to G$ is a pointwise weak equivalence, and so a pointwise acyclic cofibration, then $s : F \to X$ is a projective trivial cofibration.

Proof. First, recall that by Lemma 3.2.4, $m' : F' \to G'$ is a projective cofibration. To construct the required factorisation, we use the 2-naturality of the counit of the 2-adjoint in (10). We define $s : F \to X$ as the pushout of $m' : F' \to G'$ along $q_F : F' \to F$. The 2-naturality of the counit determines a canonical 2-natural transformation $n : X \to G$ fitting in the following diagram:



(12)

Recall that, being defined as the maps having the left lifting property with respect to the pointwise acyclic fibrations, projective cofibrations are closed under pushouts. Therefore $s : F \to X$ is a projective cofibration, since it is the pushout of $m' : F' \to G'$, which is a projective cofibration.

To show that $n : X \to G$ is a pointwise acyclic fibration, we begin by showing that it is a pointwise weak equivalence. In order to do this, we want to apply pointwise the Three-for-Two Axiom to the commuting triangle involving $q_G : G' \to G$ and $t : G' \to X$. We have already seen that q_G is a pointwise weak equivalence. For $A \in \mathcal{A}$, to show that $t_A : G'A \to XA$ is a weak equivalence, observe that it is the pushout of q_{FA} along m'_A, since pushouts in $[\mathcal{A}, \mathcal{K}]$ are also computed pointwise. But $q_{FA} : F'A \to FA$ is a weak equivalence and $m'_A : F'A \to G'A$ is a cofibration, since every projective cofibration is also a pointwise cofibration. Since every object in $\mathcal{K}$ is cofibrant, we can apply a result of Reedy [11, Proposition 13.1.2] and deduce that t_A, being the pushout of a weak equivalence between cofibrant objects along a cofibration, is a weak equivalence.

Finally, we need to show that $n : X \to G$ is a pointwise acyclic fibration, which amounts to showing that it is a pointwise surjective equivalence. This follows by the commutativity of triangle involving n and q_G in (12), since $q_G : G' \to G$ is a pointwise surjective equivalence, as shown in Lemma 3.2.3.

The second claim follows from the construction given above. First, observe that if $m : F \to G$ is a pointwise acyclic cofibration, then $m' : F' \to G'$ is a projective trivial cofibration by Lemma 3.2.4. Since $s : F \to X$ is a pushout of $m' : F' \to G'$, it inherits from m' the left lifting property with respect to the pointwise fibrations. □

We can now prove the existence of the projective model structure. We refer to axioms for a model category as Three-For-Two (Q1), Retracts (Q2), Lifting (Q3), and Factorisations (Q4). We also prove that it satisfies the additional axiom of a model 2-category, recalled in Definition 2.2.1.

3.2.6. **Theorem.** *Let $\mathcal{K}$ be a 2-category with finite limits and colimits, considered as equipped with its natural model 2-category structure. If $\mathcal{K}$ is cocomplete, then for every small 2-category $\mathcal{A}$ the 2-category $[\mathcal{A}, \mathcal{K}]$ admits the projective model structure. The projective model structure equips $[\mathcal{A}, \mathcal{K}]$ with a model 2-category structure.*

Proof. The verification of (Q1) and (Q2) is straightforward. For the rest of the proof, let us refer to a map that is both a projective cofibration and a pointwise weak equivalence as a *projective acyclic cofibration*. Also, recall that a projective trivial cofibration is a 2-natural transformation with the left lifting property with respect to pointwise acyclic fibrations.

We verify (Q4), which involves providing two factorizations. Suppose we wish to factor $m : F \to G$ as a projective cofibration followed by a pointwise acyclic fibration. First, apply Lemma 3.2.2 so as to factor m as a pointwise cofibration followed by a pointwise acyclic cofibration. Secondly, apply Lemma 3.2.5 and factor the pointwise cofibration just obtained as a projective cofibration followed by a pointwise acyclic cofibration. The projective cofibration is the first component of the required factorization, while the second is given by a composition of pointwise acyclic fibrations, which is an acyclic fibration. Next, suppose we wish to factor $m : F \to G$ as a projective acyclic cofibration followed by a pointwise fibration. First, apply Lemma 3.2.2 and factor m as a pointwise acyclic cofibration followed by a pointwise fibration. Secondly, apply Lemma 3.2.5 and factor the pointwise acyclic cofibration as a projective cofibration followed by a pointwise acyclic fibration. Now, observe that the projective cofibration is in fact a pointwise equivalence by Three-For-Two, and hence it is a projective acyclic cofibration. This provides the first part of the required factorisation. The second part is the composite of a pointwise fibration followed by a pointwise acyclic fibration, and hence it is a pointwise fibration, as required.

Finally, we prove that (Q3) holds. The first part of the statement follows by the very definition of projective cofibration. For the second part, it suffices to verify that a projective cofibration $m : F \to G$ is a pointwise weak equivalence if and only if it is a projective trivial cofibration. Let $m : F \to G$ be a projective cofibration. First, we assume that it is a pointwise weak equivalence. By Lemma 3.2.5 we can factor m as a projective trivial cofibration followed by a pointwise acyclic fibration. Next, we use the lifting property of projective cofibrations with respect to pointwise acyclic cofibrations to exhibit m as a retract of the trivial cofibration that we obtained in the factorization, which implies that m is a trivial cofibration as well. For the converse implication, let $m : F \to G$ be a trivial cofibration. Then, we can factor it as a pointwise cofibration followed by a pointwise acyclic fibration. The lifting property of projective trivial cofibrations with respect to pointwise acyclic fibrations exhibits m as a retract of the pointwise acyclic cofibration with which we factored it, and hence a pointwise weak equivalence, as required.

To check that the projective model structure is a model 2-category structure, we use part (*ii*) of Lemma 2.2.2. Since cotensors in diagram 2-categories are computed pointwise, the required statement follows from the corresponding fact for $\mathcal{K}$. □

3.2.7. *Remark.* As an instance of the general definition of flexible algebra for a 2-monad [4], a diagram F is said to be *flexible* if $q_F : F' \to F$ has a section in $[\mathcal{A}, \mathcal{K}]$. As a special case of [17, Theorem 4.12] we obtain that F is flexible if and only if it is projectively cofibrant. The 2-adjunction in (10) determines a 2-comonad on $[\mathcal{A}, \mathcal{K}]$ whose underlying 2-functor $Q : [\mathcal{A}, \mathcal{K}] \to [\mathcal{A}, \mathcal{K}]$ is defined by letting $QF = (JF)'$. The counit of the 2-adjunction is the counit of the 2-comonad, and so we can write its components as $q_F : QF \to F$. Since every 2-functor of the form QF is projectively cofibrant, the acyclic fibrations $q_F : QF \to F$ provide 2-functorial cofibrant replacements for the projective model structure. Comonadic cofibrant replacements are studied in [6].

4. Injective model structures on 2-categories of diagrams

4.1. **The dual of the natural model structure on a 2-category.** The notion of equivalence being self-dual, a map is an equivalence in $\mathcal{K}$ if and only if it is an equivalence in $\mathcal{K}^{\mathrm{op}}$. We say that a map is an *isocofibration* in $\mathcal{K}$ if it is an isofibration in $\mathcal{K}^{\mathrm{op}}$. By duality, there exists a model 2-structure on $\mathcal{K}$ in which the weak equivalences are the equivalences and the cofibrations are the isocofibrations. We refer to this model structure as the *dual of the natural model structure* on $\mathcal{K}$. The acyclic cofibrations in the dual of the model structure are the injective equivalences. Its existence can be readily deduced from Steve Lack's general theorem concerning the existence of the natural model structure: the weak equivalences, fibrations, and cofibrations in the dual of the natural model structure on $\mathcal{K}$ are respectively the weak equivalences, cofibrations, and fibrations in the natural model structure on $\mathcal{K}^{\mathrm{op}}$. The fact that the dual of the natural model structure is a 2-model structure follows easily from the fact that the natural model structure is so.

We wish to relate the natural model structure on $\mathcal{K}$ and its dual. Let us write J for the category with two objects and an isomorphism between them. We can identify functors $J \to \mathcal{K}(X, B)$ with the data of a pair of maps $b_0 : X \to B$, $b_1 : X \to B$ and an invertible 2-cell $\beta : b_0 \to b_1$ in $\mathcal{K}$. For $X \in \mathcal{K}$, the universal property of tensors in (3) implies that the J-tensor of X, written $J \otimes X$, comes equipped with a canonical functor $e : J \to \mathcal{K}(X, J \otimes X)$ which induces by composition a natural isomorphism of categories with components $\mathcal{K}(J \otimes X, B) \cong \mathbf{Cat}(J, \mathcal{K}(X, B))$. The map $e_0 : X \to J \otimes X$ is an injective equivalence, and hence an acyclic cofibration in the dual of the natural model structure on $\mathcal{K}$.

4.1.1. **Lemma.** *Let $\mathcal{K}$ be a 2-category with finite limits and colimits.*

(*i*) *If a map has the right lifting property with respect to the injective equivalences, then it is an isofibration.*

(*ii*) *If a map has the left lifting property with respect to the surjective equivalences, then it is an isocofibration.*

Proof. First, observe that (*i*) and (*ii*) are equivalent by duality. We prove (*ii*). Let $f : A \to B$ be a map with the right lifting property with respect to injective equivalences. Given a diagram as in (5), the 2-cell $\beta : b \Rightarrow fa$ induces a functor $J \to \mathcal{K}(X, B)$. By the universal property of $J \otimes X$, we obtain a map $\langle\beta\rangle : J \otimes X \to B$ making the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ {\scriptstyle e_0}\downarrow & & \downarrow{\scriptstyle f} \\ J \otimes X & \xrightarrow[\langle\beta\rangle]{} & B \end{array}$$

Since $f : A \to B$ has the right lifting property with respect to injective equivalences and $e_0 : X \to J \otimes X$ is an injective equivalence, there is a filler for the diagram, which can be used to construct the appropriate 1-cells and 2-cells for the diagram in (6), thus showing that $f : A \to B$ is an isofibration, as required. $\square$

4.1.2. **Proposition.** *The identity* 2*-functor induces a Quillen* 2*-equivalence between the natural model structure and its dual.*

Proof. We consider the identity 2-functor as going from $\mathcal{K}$ equipped with its natural model structure to $\mathcal{K}$ equipped with the dual of the natural model structure. We wish to show that it is left Quillen 2-functor. The identity 2-functor clearly preserves weak equivalences. Since the cofibrations in the natural model structure are the maps with the left lifting property with respect to the surjective equivalences, and the cofibrations in the dual of the natural model structure are the isocofibrations, part (*ii*) of Lemma 4.1.1 shows that the identity preserves cofibrations. Hence the identity 2-functor preserves weak equivalences and acyclic cofibrations, as required. $\square$

The natural model structure on **Cat** and its dual not only are Quillen 2-equivalent, but actually coincide.

4.2. **The injective model structure.** Let $\mathcal{A}$ be a small 2-category and $\mathcal{K}$ be a model 2-category. A 2-natural transformation between diagrams is said to be an *injective fibration* if it has the right lifting property with respect to the pointwise acyclic cofibrations. Relative to the model 2-structure on $\mathcal{K}$, we define the injective model structure on $[\mathcal{A}, \mathcal{K}]$ as follows:

$$[\mathcal{A}, \mathcal{K}]_{\text{Inj}} = \begin{cases} \text{Weak equivalences} & = \text{pointwise weak equivalences,} \\ \text{Fibrations} & = \text{injective fibrations,} \\ \text{Cofibrations} & = \text{pointwise cofibrations.} \end{cases}$$

By duality, Theorem 3.2.6 implies Corollary 4.2.1.

4.2.1. **Corollary.** *Let $\mathcal{K}$ be a* 2*-category with finite limits and colimits, considered as equipped with the dual of its natural model* 2*-category structure.*

If $\mathcal{K}$ is complete, then for every small 2-category $\mathcal{A}$ the 2-category $[\mathcal{A},\mathcal{K}]$ admits the injective model structure. The injective model structure equips $[\mathcal{A},\mathcal{K}]$ with a model 2-category structure.

Proof. The 2-category $[\mathcal{A},\mathcal{K}]$ can be identified with the 2-category $[\mathcal{A}^{\mathrm{op}},\mathcal{K}^{\mathrm{op}}]$. Considering the dual of the natural model structure $\mathcal{K}$ and the injective model structure on $[\mathcal{A},\mathcal{K}]$ is the same as considering the natural model structure on $\mathcal{K}^{\mathrm{op}}$ and the projective model structure on $[\mathcal{A}^{\mathrm{op}},\mathcal{K}^{\mathrm{op}}]$. The latter exists by Theorem 3.2.6. □

4.2.2. *Remark.* Duality can be used also to observe that the inclusion 2-functor in (10) admits not only a left 2-adjoint but also a right 2-adjoint:

$$[\mathcal{A},\mathcal{K}] \underset{(\,\cdot\,)^*}{\overset{J}{\rightleftarrows}} \mathrm{Psd}[\mathcal{A},\mathcal{K}] \qquad (13)$$

The unit of the 2-adjunction in (13) has components given by 2-natural transformations $r_F : F \to F^*$ that are pointwise injective equivalences, and so pointwise acyclic cofibrations in the injective model structure. The 2-adjunction in (13) determines a 2-monad on $[\mathcal{A},\mathcal{K}]$ whose underlying 2-functor $R : [\mathcal{A},\mathcal{K}] \to [\mathcal{A},\mathcal{K}]$ is defined by letting $RF = J(F^*)$. Here, the unit of the 2-monad provides 2-functorial fibrant replacements for the injective model structure, since its components $r_F : F \to RF$ are acyclic cofibrations in the injective model structure, and each RF is injectively fibrant. When $\mathcal{A}$ is a category and $\mathcal{K}$ is **Cat**, we will give explicit formulas for RF in Section 6.4.

We conclude this section by proving that the projective and the injective model structures are Quillen 2-equivalent.

4.2.3. **Lemma.** *Let $\mathcal{A}$ be a small 2-category. Let $\mathcal{K}$ be a 2-category with finite limits and colimits.*

- (*i*) *If a 2-natural transformation has the right lifting property with respect to pointwise injective equivalences, then it is a pointwise isofibration.*
- (*ii*) *If a 2-natural transformation has the left lifting property with respect to surjective equivalences, then it is a pointwise isocofibration.*

Proof. We prove (*i*). Let $m : F \to G$ be a 2-natural transformation and assume that it has the the right lifting property with respect to pointwise injective equivalences. By part (*i*) of Lemma 4.1.1, to prove that it is a pointwise isofibration, it is sufficient to prove that every component $m_A : FA \to GA$ has the left lifting property with respect to injective equivalences in $\mathcal{K}$. But this follows immediately since $m : F \to G$ has the right lifting property with respect to pointwise injective equivalences. □

4.2.4. **Proposition.** *The identity 2-functor induces a Quillen 2-equivalence between the projective and the injective model structure.*

Proof. The claim follows from Lemma 4.2.3 by the same reasoning used in the proof of Proposition 4.1.2. □

Although the projective and injective model structures are Quillen 2-equivalent, we will see in Section 5 and Section 6 that they play very different roles in the study of homotopy limits.

5. Model structures for homotopy limits

5.1. **Quillen 2-adjunctions in two variables.** To study homotopy limits for 2-categories, we use a straighforward 2-categorical analogue of the notion of a Quillen adjunction in two variables [12]. A 2-functor of the form $\Phi : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$, where $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{M}$ are 2-categories, will be referred to as a 2-*functor in two variables*. Given $\Phi : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$, for $u : X \to Y$ in $\mathcal{K}$ and $v : V \to W$ in $\mathcal{L}$, we have the commutative diagram

$$\begin{array}{ccc} \Phi(X,V) & \xrightarrow{\Phi(u,V)} & \Phi(Y,V) \\ {\scriptstyle \Phi(X,v)}\downarrow & & \downarrow{\scriptstyle \Phi(Y,v)} \\ \Phi(X,W) & \xrightarrow[\Phi(u,W)]{} & \Phi(Y,W) \end{array}$$

When $\mathcal{M}$ has pushouts, the commutativity of the diagram determines a canonical map

$$\langle u, v \rangle : \Phi(X,V) \sqcup_{\Phi(X,U)} \Phi(Y,U) \to \Phi(Y,V)\,. \tag{14}$$

When the underlying categories of $\mathcal{K}, \mathcal{L}, \mathcal{M}$ are equipped with model structures, we say that $\Phi : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ is a *left Quillen* 2*-functor in two variables* if Φ is cocontinuous in each variable, and if $u : X \to Y$ is a cofibration in $\mathcal{K}$ and $v : U \to V$ is a cofibration in $\mathcal{L}$, then $\langle u, v \rangle$ is a cofibration in $\mathcal{M}$, which is also a weak equivalence when either u or v is so. We say that $\Phi : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ is a *right Quillen* 2*-functor in two variables* if its dual $\Phi^{\mathrm{op}} : \mathcal{K}^{\mathrm{op}} \times \mathcal{L}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$ is a left Quillen functor in two variables. Quillen 2-functors in two variables often arise in a special kind of situation, which is convenient to isolate. Recall that a 2-adjunction in two variables consists of 2-functors

$$\Phi : \mathcal{K} \times \mathcal{L} \to \mathcal{M}\,, \quad \Theta : \mathcal{L}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{K}\,, \quad \Psi : \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{L}\,,$$

and 2-natural isomorphisms, for $X \in \mathcal{K}$, $Y \in \mathcal{L}$, and $Z \in \mathcal{M}$

$$\mathcal{K}\big(X, \Theta(Y,Z)\big) \cong \mathcal{M}\big(\Phi(X,Y), Z\big) \cong \mathcal{L}\big(Y, \Psi(X,Z)\big)\,,$$

In these circumstances, Φ is a left 2-adjoint in two variables, while Ψ and Θ are right 2-adjoints in two variables. This notion of two-variable adjunction for enriched categories has been studied in [9]. When the underlying categories of $\mathcal{K}, \mathcal{L}$ and $\mathcal{M}$ are equipped with model structures we have a *Quillen* 2*-adjunction in two variables* if the following equivalent conditions hold:

(i) $\Phi : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ is left Quillen 2-functor in two variables,
(ii) $\Theta : \mathcal{L}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{K}$ is right Quillen 2-functor in two variables,

(iii) $\Psi : \mathcal{K}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{K}$ is right Quillen 2-functor in two variables.

The equivalence between these conditions is in [12, Lemma 4.2.2]. For example, When $\mathcal{K}$ is tensored and cotensored, the 2-adjunctions in (3) and (4) allow us to obtain a 2-adjunction in two variables involving the following 2-functors:

$$\begin{aligned} \Phi &: \mathbf{Cat} \times \mathcal{K} \to \mathcal{K}\,, & \Phi(A,X) &=_{\mathrm{def}} A \otimes X \\ \Theta &: \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathbf{Cat}\,, & \Theta(X,Y) &=_{\mathrm{def}} \mathcal{K}(X,Y) \\ \Psi &: \mathbf{Cat}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{K}\,, & \Psi(A,Y) &=_{\mathrm{def}} Y^A\,. \end{aligned}$$

Lemma 2.2.2 can then be rephrased as follows.

5.1.1. **Lemma.** *Let $\mathcal{K}$ be a 2-category with finite limits and colimits, whose underlying category is equipped with a model structure. When $\mathcal{K}$ is tensored and cotensored, $\mathcal{K}$ is a model 2-category if and only the following equivalent conditions hold.*

(i) The tensor 2-functor is a left Quillen 2-functor in two variables.
(ii) The hom 2-functor is a right Quillen 2-functor in two variables.
(iii) The cotensor 2-functor is a right Quillen 2-functor in two variables.

5.2. **Weighted limits.** Existence of conical 2-limits in a 2-category $\mathcal{K}$ can be expressed by saying that for every small 2-category $\mathcal{A}$ we have a 2-adjunction of form

$$\mathcal{K} \underset{\lim}{\overset{\Delta}{\underset{\bot}{\rightleftarrows}}} [\mathcal{A}, \mathcal{K}]\,. \tag{15}$$

The left 2-adjoint is the diagonal 2-functor, and the right 2-adjoint sends a diagram to its limit. When $\mathcal{K}$ is equipped with the dual of its natural model structure and $[\mathcal{A}, \mathcal{K}]$ with the injective model structure, we obtain a Quillen 2-adjunction, since the diagonal 2-functor clearly preserves weak equivalences and cofibrations. As an enriched category, however, $\mathcal{K}$ admits more general notions of limits, known as weighted limits [16, 22], whose homotopy-theoretic behaviour of weighted limits is less straighforward.

To recall weighted limits, we refer to 2-functors $J : \mathcal{A} \to \mathbf{Cat}$ as *weights*. Existence of weighed limits for a fixed diagram F can be expressed as the existence of a 2-adjunction of the form

$$\mathcal{K} \underset{\{-,F\}}{\overset{\mathcal{K}(\Delta(-),F)}{\underset{\bot}{\rightleftarrows}}} [\mathcal{A}, \mathbf{Cat}]^{\mathrm{op}} \tag{16}$$

The left 2-adjoint sends $X \in \mathcal{K}$ into the weight $\mathcal{K}(X, F(-)) : \mathcal{A} \to \mathbf{Cat}$, while the right 2-adjoint sends a weight $J : \mathcal{A} \to \mathbf{Cat}$ to the J-weighted limit of F, denoted $\{J, F\}$ as usual [15]. Thus, the J-weighted limit of F is characterized by 2-natural isomorphisms of form

$$[\mathcal{A}, \mathbf{Cat}]\big(J(-), \mathcal{K}(X, F(-))\big) \cong \mathcal{K}\big(X, \{J, F\}\big)\,.$$

When $\mathcal{K}$ is tensored, for a fixed weight $J : \mathcal{A} \to \mathbf{Cat}$, existence of J-weighted limits can be expressed equivalently as the existence of a 2-adjunction of form

$$\mathcal{K} \underset{\{J,-\}}{\overset{J\otimes\Delta(-)}{\underset{\bot}{\rightleftarrows}}} [\mathcal{A}, \mathcal{K}] \tag{17}$$

The left 2-adjoint should be thought of as 'J-weighted diagonal': it sends $X \in \mathcal{K}$ into the diagram $J(-) \otimes X : \mathcal{A} \to \mathcal{K}$. The right 2-adjoint sends an diagram to its J-weighted limit. Therefore, we can characterize $\{J, F\}$ also by the existence of 2-natural isomorphism with components

$$[\mathcal{A}, \mathcal{K}]\big(J(-) \otimes X, F(-)\big) \cong \mathcal{K}\big(X, \{J, F\}\big) .$$

It should be noted how the 2-adjunction in (15) is analogous to that in (17). Writing $\mathbf{1} : \mathcal{A} \to \mathbf{Cat}$ for the weight with constant value the terminal category, it is immediate to see that weighted limits subsume conical limits [15, §3.8], since the limit of a diagram F can be viewed as the weighted limit by the existence of an isomorphism

$$\lim(F) \cong \{\mathbf{1}, F\} .$$

The map assigning to a weight and a diagram the corresponding weighted limit determines a 2-functor in two variables. By the 2-adjunctions in (16) and (17), the weighted limit 2-functor is a right 2-adjoint in two variables. It follows that there is a 2-adjunction in two variables involving the following 2-functors:

$$\Phi : [\mathcal{A}, \mathbf{Cat}] \times \mathcal{K} \to [\mathcal{A}, \mathcal{K}] , \qquad \Phi(J, X) =_{\text{def}} J(-) \otimes X \tag{18}$$

$$\Theta : \mathcal{K}^{\text{op}} \times [\mathcal{A}, \mathcal{K}] \to [\mathcal{A}, \mathbf{Cat}] , \qquad \Theta(X, F) =_{\text{def}} \mathcal{K}(X, F(-)) \tag{19}$$

$$\Psi : [\mathcal{A}, \mathbf{Cat}]^{\text{op}} \times [\mathcal{A}, \mathcal{K}] \to \mathcal{K} , \qquad \Psi(J, F) =_{\text{def}} \{J, F\} . \tag{20}$$

From now on, we assume that $\mathcal{K}$ is complete and cocomplete, so that we have both the projective and the injective model structure. We establish that there are two choices of model structures on functor 2-categories that allow us to regard the weighted limit 2-functor as a right Quillen 2-functor in two variables, and so the 2-adjunction in two variables above as a Quillen 2-adjunction in two variables. The first possibility is stated in Proposition 5.2.1.

5.2.1. **Proposition.** *Considering both the* 2*-category of weights* $[\mathcal{A}, \mathbf{Cat}]$ *and the* 2*-category of diagrams* $[\mathcal{A}, \mathcal{K}]$ *as equipped with the projective model structure, the weighted limit* 2*-functor is a right Quillen* 2*-functor.*

Proof. It suffices to verify that the 2-functor Θ defined in (19) is a right Quillen functor in two variables. This follows from the fact that the hom-category 2-functor is so, as stated in Lemma 5.1.1. □

As stated in Proposition 5.2.2, there exists a second choice of Quillen model structures that makes the weighted limit 2-functor into a right Quillen 2-functor in two variables.

5.2.2. **Proposition.** *Considering both the* 2*-category of weights* $[\mathcal{A}, \mathbf{Cat}]$ *and the* 2*-category of diagrams* $[\mathcal{A}, \mathcal{K}]$ *as equipped with the injective model structure, the weighted limit* 2*-functor is a right Quillen* 2*-functor in two variables.*

Proof. It is sufficient to establish that the 2-functor Φ defined in (18) is a left Quillen 2-functor in two variables. To prove this, it is sufficient to recall that the dual of the natural model structure is a model 2-structure and so, by Lemma 5.1.1, the tensor functor for $\mathcal{K}$ is a left Quillen 2-functor in two variables. □

5.2.3. *Remark.* There are counterparts of these statements for weighted colimits, which we prefer to avoid stating to avoid repetition. To make the weighted colimit functor into a left Quillen 2-functor in two variables there are again two possible choices of model structures. The first involves the projective model structure on weights and the injective model structure on diagrams; the second involves the injective model structure on weights and the projective model structure on diagrams. The development in Section 6 has an evident analogue for weighted colimits.

6. Homotopy limits

6.1. **Weighted homotopy limits.** Proposition 5.2.1 and Proposition 5.2.2 allow us to apply the general theory of model $\mathcal{V}$-categories [7, 12] and conclude the existence of the total derived $\mathrm{Ho}(\mathbf{Cat})$-enriched functor of the weighted limit 2-functor. We refer to it as the *weighted homotopy limit* functor. We relate the notion of a weighted homotopy limit to that of a *weighted pseudo-limit* [3, 16, 22]. The latter is obtained by replacing 2-natural transformations with pseudo-natural transformations in the 2-categories of weights and diagrams involved in the definition of weighted limit. For example, the existence of conical pseudo-limits in $\mathcal{K}$ is expressed by saying that for every small 2-category $\mathcal{A}$ we have a 2-adjunction of form

$$\mathcal{K} \underset{\text{pslim}}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Psd}[\mathcal{A}, \mathcal{K}]\,. \tag{21}$$

Here the left 2-adjoint is obtained by composing the diagonal 2-functor in (17) with the inclusion 2-functor in (8). Note that, even if we are considering pseudo-natural transformations, we are still requiring that (21) is a 2-adjunction. Therefore, the pseudo-limit of a diagram F is characterized by a 2-natural isomorphism

$$\mathrm{Psd}[\mathcal{A}, \mathcal{K}](\Delta X, F) \cong \mathcal{K}(X, \mathrm{pslim}\, F)\,.$$

Note that a pseudo-natural transformation $\Delta X \to F$ can be seen as a cone commuting up to coherent isomorphism. We write $\{J, F\}_p$ for the J-weighted pseudo-limit of F, which can be defined analogously. The connection between weighted homotopy limits and weighted pseudo-limits follows from the observation that, as a special case of the analysis in [17,

§4.14], the homotopy Ho(**Cat**)-category of the model 2-categories $[\mathcal{A}, \mathcal{K}]$ can be identified with the Ho(**Cat**)-category associated to the 2-category $\mathrm{Psd}[\mathcal{A}, \mathcal{K}]$ by the localization functor $\lambda : \mathbf{Cat} \to \mathrm{Ho}(\mathbf{Cat})$. This implies that a weighted pseudo-limit, when regarded as an object of the homotopy Ho(**Cat**)-category, is a weighted homotopy limit.

6.2. **Two resolutions.** We wish to show how the two different ways of expressing homotopy weighted limits as weighted limits suggested by Proposition 5.2.1 and Proposition 5.2.2 correspond exactly to two ways of expressing weighted pseudo-limits as weighted limits. If we consider both the 2-category of diagrams and the 2-category of weights as equipped with the projective model structure, Proposition 5.2.1 leads us to define the total right derived functor of the weighted limit 2-functor by letting

$$\{J, F\}^{\mathrm{R}} =_{\mathrm{def}} \{QJ, F\} \tag{22}$$

Here QJ denotes the cofibrant replacement of the weight J with respect to the projective model structure, as in Remark 3.2.7. Note that it is not necessary to replace F since every diagram is projectively fibrant. This formula is closely related to a well-known resul showing that the existence of weighted limits implies the existence of weighted pseudo-limits [3, 16]. Indeed, using the left 2-adjoint in (10), we have the sequence of 2-natural isomorphisms

$$\begin{aligned} \mathcal{K}(X, \{J, F\}_p) &\cong \mathrm{Psd}[\mathcal{A}, \mathbf{Cat}]\big(J, \mathcal{K}(X, F(-))\big) \\ &\cong [\mathcal{A}, \mathbf{Cat}]\big(QJ, \mathcal{K}(X, F(-))\big) \\ &\cong \mathcal{K}(X, \{QJ, F\}) \end{aligned}$$

The Yoneda lemma for 2-categories implies the existence of an isomorphism

$$\{J, F\}_p \cong \{QJ, F\}\,. \tag{23}$$

There is a different, but equivalent, definition for the total right derived functor of the weighted limit 2-functor, which follows from Proposition 5.2.2. If we consider the weighted limit 2-functor as a Quillen 2-functor in two variables with respect to the injective model structures, its total right derived Ho(**Cat**)-enriched functor can be defined by letting

$$\{J, F\}^{\mathrm{R}} =_{\mathrm{def}} \{J, RF\}\,. \tag{24}$$

Here $RF : \mathcal{A} \to \mathcal{K}$ denotes the fibrant replacement of a diagram $F : \mathcal{A} \to \mathcal{K}$ with respect to the injective model structure, as in Remark 4.2.2. Note that we do not need to make any replacement for J, since any weight is injectively cofibrant. This corresponds to a different way of reducing pseudo-limits to weighted limits, which does not seem to appear in the existing literature. Assuming that $\mathcal{K}$ has tensors, and using the right 2-adjoint in (13), we have the sequence of 2-natural isomorphisms

$$\begin{aligned}\mathcal{K}(X, \{J, F\}_p) &\cong \mathrm{Psd}[\mathcal{A}, \mathcal{K}]\big(J(-) \otimes X, F(-)\big) \\ &\cong [\mathcal{A}, \mathcal{K}]\big(J(-) \otimes X, RF(-))\big) \\ &\cong \mathcal{K}(X, \{J, RF\})\end{aligned}$$

Hence, we conclude as before that there is an isomorphism

$$\{J, F\}_p \cong \{J, RF\}\,. \tag{25}$$

6.3. **Homotopy limits.** The two choices of model structures making the weighted limit functor into a right Quillen 2-functor can be used also in the computation of homotopy limits. Let us now consider $\mathcal{K}$ as being equipped with its natural model 2-structure. We can therefore define the homotopy limit of a diagram F by letting

$$\mathrm{holim}(F) =_{\mathrm{def}} \{\mathbf{1}, F\}^{\mathrm{R}} \tag{26}$$

A first way to compute the homotopy limit is to apply the formula in (22) and obtain

$$\mathrm{holim}\, F = \{Q\mathbf{1}, F\}$$

This is in fact a consequence of the well-known formula [3, 16, 22] expressing the pseudo-limit of a diagram $F : \mathcal{A} \to \mathcal{K}$ as the weighted limit $\{Q\mathbf{1}, F\}$. The formula has at least two noteworthy aspects. First, it involves only the projective model structures. Hence, in the context of model 2-categories, homotopy limit 2-functors can be defined without injective model structures. Secondly, $Q\mathbf{1}$ is the cofibrant replacement of the constant weight $\mathbf{1} : \mathcal{A} \to \mathbf{Cat}$ with respect to the projective model structure on $[\mathcal{A}, \mathbf{Cat}]$. Even if cofibrant replacements with respect to projective model structures are generally rather involved, the very simple nature of the weight simplifies the task considerably. If we apply the formula in (24), instead, we obtain an isomorphism

$$\mathrm{holim}\, F \cong \lim RF\,.$$

This is the standard homotopy-theoretic formula for the homotopy limit of a diagram, which involves considering the injective model structure on $[\mathcal{A}, \mathcal{K}]$ and computing the right derived functor of the limit functor. Remarkably, this corresponds to a reduction of pseudo-limits to conical 2-limits which does not seem to appear in the existing literature. Indeed, using the right 2-adjoint in (13), we get

$$\begin{aligned}\mathcal{K}(X, \mathrm{pslim}\, F) &= \mathrm{Psd}[\mathcal{A}, \mathcal{K}](\Delta X, F) \\ &= [\mathcal{A}, \mathcal{K}](\Delta X, RF) \\ &\cong \mathcal{K}(X, \lim RF)\,.\end{aligned}$$

Hence, we obtain an isomorphism

$$\mathrm{pslim}\, F \cong \lim RF\,.$$

This isomorphism, which is a special case of (25), reduces pseudo-limits to conical 2-limits. The assumption that $\mathcal{K}$ is complete and cocomplete is necessary to do so. Indeed, as discussed in [3, §2], the full sub-2-category of **Cat** whose objects are the categories with at most one object has all conical limits, but not all pseudo-limits.

6.4. **Some formulas.** When $\mathcal{A}$ is an ordinary category and $\mathcal{K}$ is **Cat**, it is possible to provide some explicit formulas for the 2-monad and 2-comonad providing the fibrant replacement and cofibrant replacement in the injective and projective model structures, respectively, discussed in Remark 3.2.7 and Remark 4.2.2. For notational convenience, we prefer to consider controvariant functors. The right 2-adjoint admits a simple description, which is determined by the Yoneda Lemma for 2-categories. For $A \in \mathcal{A}$, an object of F^*A can be identified with a 2-natural transformation $\mathbf{y}(A) \to F^*$, where $\mathbf{y}(A) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ denotes the Yoneda embedding of A. By 2-adjointness, this 2-natural transformation should correspond to a pseudo-natural transformation $\mathbf{y}(A) \to F$. Hence, we are led to define the 2-monad for fibrant replacements as

$$RF(A) =_{\mathrm{def}} \mathrm{Psd}[\mathcal{A}, \mathbf{Cat}](\mathbf{y}(A), F)\,.$$

The right-hand side can be described equivalently in terms of cartesian sections of Grothendieck fibrations, as in [8, §I.2.4.4.1].

A formula for the 2-comonad follows by a direct application of the results in [8, §I.2.4]. We write $\mathbf{Fib}(\mathcal{A})$ for the 2-category of Grothendieck fibrations over $\mathcal{A}$, cartesian functors, and fibred natural transformations [8, §I.1.8]. The familiar Grothendieck construction [10, §VI.8] provides a 2-functor $\mathrm{Tot} : [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{Fib}(\mathcal{A})$ mapping a functor $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ into a split Grothendieck fibration $\mathrm{Tot}\, F \to \mathcal{A}$. By [8, §I.2.4.3] this 2-functor has a left 2-adjoint $L : \mathbf{Fib}(\mathcal{A}) \to [\mathcal{A}, \mathbf{Cat}]$. Hence we obtain the following isomorphisms

$$[\mathcal{A}, \mathbf{Cat}]\big(L(\mathrm{Tot}\, F), G\big) \cong \mathbf{Fib}(\mathcal{A})\big(\,\mathrm{Tot}\, F, \mathrm{Tot}\, G\big) \cong \mathrm{Psd}[\mathcal{A}, \mathbf{Cat}]\big(F, G\big)$$

The first isomorphisms follows by the 2-adjointness $L \dashv \mathrm{Tot}$, while the second follows by the well-known identification between pseudo-natural transformations $F \to G$ and cartesian functors $\mathrm{Tot}\, F \to \mathrm{Tot}\, G$. We can therefore define the 2-comonad for cofibrant replacements by letting

$$QF =_{\mathrm{def}} L(\mathrm{Tot}\, F)\,.$$

We wish to undwind this definition. For $A \in \mathcal{A}$, let us write $\mathcal{A}^{\backslash A}$ for the category whose objects are the arrows in $\mathcal{A}$ with domain A, and with maps the evident commuting triangles. There is then a canonical functor $\mathcal{A}^{\backslash A} \to \mathcal{A}$ mapping an arrow into its codomain, and we write $\mathrm{Tot}(F)^{\backslash A} \to \mathcal{A}^{\backslash A}$ for the Grothendieck fibration obtained by pulling back the Grothendieck fibration $\mathrm{Tot}\, F \to \mathcal{A}$ along it. By the definitions in [8, §I.2.4.3] we obtain

$$QF(A) = \varinjlim(\mathrm{Tot}(F)^{\backslash A}/\mathcal{A}^{\backslash A})\,,$$

The right-hand side denotes the category obtained by localizing the category $\mathrm{Tot}(F)^{\backslash A}$ with respect to the set of its cartesian morphisms, with universal properties as in [2, §VI.6].

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